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Final Project Presentations

Vote your favorites. The most votes win fabulous non-cash prizes!

1

Quiz 0

Wed, 18 Jan 2023

1. What is your preferred name for in-class use?

Solution

Michael, Professor Jahn, Prof Jahn, Prof, and Dr. Jahn, are all ok ♠♥♣♦

2. Interests (academic and non-):

Solution

Many mathematical interests. Photography. ♠♥♣♦

3. Star Trek or Star Wars (circle one)

Solution

Class: Star Trek: 3, Star Wars: 22
Professor: Star Trek
Star Trek WINS !!! ♠♥♣♦

4. What is the best day(s) for Office Hours?

Solution

All ok for me. ♠♥♣♦

5. Math background: If a graph G has 10 vertices, what is the max number of triangles in G ?

Solution

$\binom{10}{3} = 120$ ♠♥♣♦

6. Math background: What is the difference between a basis and a spanning set?

Solution

(Linear algebra) A basis is a *linearly independent* spanning set. Most spanning sets are not linearly indep. ♠♥♣♦

7. Math background: What is your favorite equivalence relation in CS?

Solution

It's a toss up: Graph isomorphism or Turing equivalence of sets of integers
(both in the context of complexity theory, so somewhat theoretical-mathy computer science).

♠♥♣♦

8. Math background: If a non-technical person asks “what is a proof”, how would you answer?

Solution

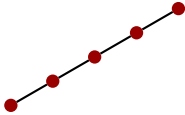
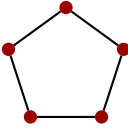
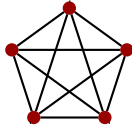
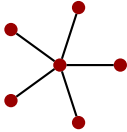
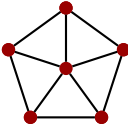
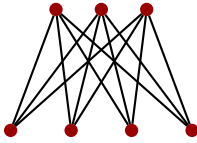
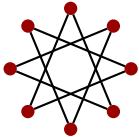
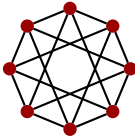
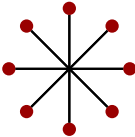
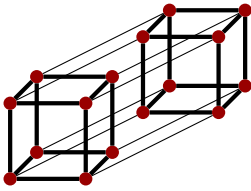
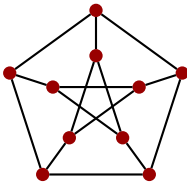


~~A mystical magical invocation of Thoth, meant only for the elect.~~

A logically airtight, logically complete, *explanation* (for why some mathematical claim is true)

♠♥♣♦

Some standard named graphs

 <p>Path P_5</p>	 <p>Cycle C_5</p>	 <p>Complete K_5</p>
 <p>Star $S_6 = K_{1,5}$</p>	 <p>Wheel W_6</p>	 <p>Complete bipartite $K_{3,4}$</p>
 <p>Circulant $\text{Circ}_8(3)$</p>	 <p>Circulant $\text{Circ}_8(1,3)$</p>	 <p>Circulant $\text{Circ}_8(4)$</p>
 <p>Hypercube Q_4</p>	 <p>Petersen</p>	

For a list of more (and more “sophisticated”) examples, go to https://en.wikipedia.org/wiki/Gallery_of_named_graphs. And you can check the `tkz-berge` documentation.

Homework 1

Due: Fri, 27 Jan 2023

1. **Problem 1.1.16.** Determine whether the two graphs below are isomorphic.

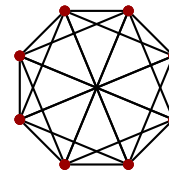
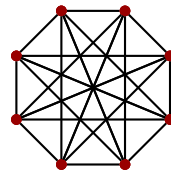
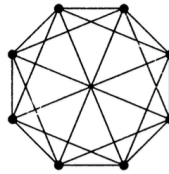
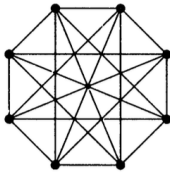


Figure 1: Import screenshots

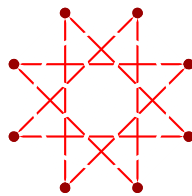
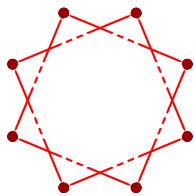
Figure 2: Create images with tikz

Key from text:
 (-) = easier,
 (+) = harder,
 (!) = useful or
 instructive,
 (*) = uses optional
 material

Solution

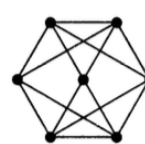
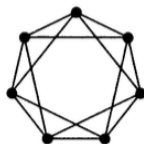
Put your solution here. Please make it be blue 😊. You can also choose your delimiter: ♠ = "it is what it is", ♥ = "I love it", ♣ = "it beat me up", ♦ = "it's a gem". Note that they are not mutually exclusive.

For this problem, the easiest way is to use `tikz` to autorender the complements. First draw a complete graph in red, then draw the original on top of the K_8 in white. This will leave only the complement. We easily see the graphs are not isomorphic because their complements are not (e.g. one is two 4-cycles, the other is a C_8).



♠♥♣♦

2. **Problem 1.1.22.(!)** Determine which pairs of graphs below are isomorphic, presenting the proof by testing the smallest number of pairs.



3. **Problem 1.1.23.** In each class below, determine the smallest n such that there exist nonisomorphic n -vertex graphs having the same vertex degree sequence.

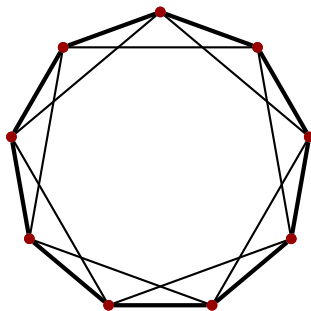
(a) all graphs, (b) loopless graphs, (c) simple graphs

4. **Problem 1.1.25.** (!) Prove the Petersen graph has no cycle of length 7.

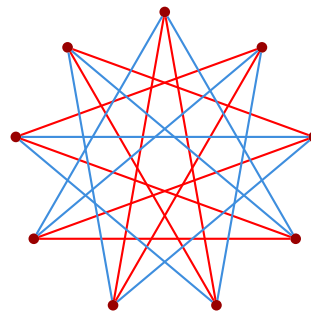
5. **Problem 1.1.33.** For $n = 5$, $n = 7$, and $n = 9$, decompose K_n into copies of C_n .

Solution

$n = 9$: Note that K_9 decomposes into the four 9-cycles in several ways. Here is one different from what the solution manual gives:

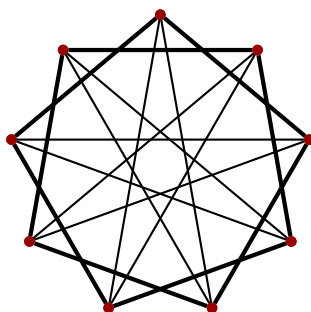


$\text{Circ}_9(1) \cup \text{Circ}_9(2)$

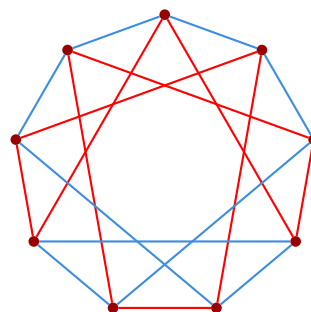


$(0, 3, 7, 2, 6, 1, 4, 8, 5) \cup (0, 4, 7, 1, 5, 2, 8, 3, 6)$

Here's another decomposition different from the solution manual.

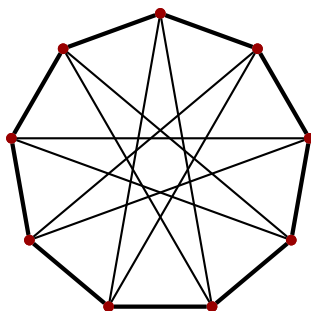


$\text{Circ}_9(2) \cup \text{Circ}_9(4)$

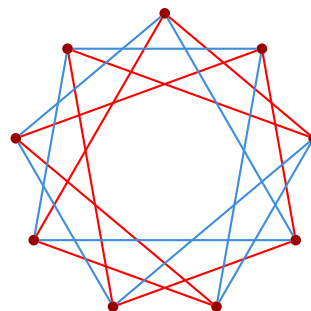


$(0, 3, 6, 7, 1, 4, 5, 2, 8, 0) \cup (0, 1, 2, 3, 4, 7, 8, 5, 6)$

Here's yet another decomposition different from the solution manual.



$\text{Circ}_9(1) \cup \text{Circ}_9(4)$



$(0, 3, 6, 8, 1, 4, 7, 5, 2, 0) \cup (0, 7, 1, 3, 5, 8, 2, 4, 6, 0)$

Note that all of these are symmetric about some axis.



6. **Problem 1.1.35.** (!) Prove that K_n decomposes into three pairwise-isomorphic subgraphs if and only if $n + 1$ is not divisible by 3. (Hint: For the case where n is divisible by 3, split the vertices into three sets of equal size.)
7. **Problem 1.1.38.** (!) Let G be a simple graph in which every vertex has degree 3. Prove that G decomposes into claws if and only if G is bipartite.

Quiz 1

Mon, 30 Jan 2023

1. Both of the following graphs decompose into a C_7 and a P_7 . Are they isomorphic? Explain.



Solution

No, the first complement has a 4-cycle, the second does not.

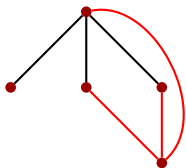
♠♥♣♦

2. Prove or disprove: Any simple graph G that decomposes into claws (i.e., into copies of $K_{1,3}$), is bipartite.

Circle which you are doing. Then do it.

Solution

Disprove with a counterexample (we need 3-regularity for the above to be true): This graph has a 3-cycle, so is not bipartite.

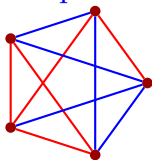


♠♥♣♦

3. Give a simple graph G such that $G \not\cong \overline{G}$, but G, \overline{G} have the same degree sequence. Use as few vertices as possible.

Solution

From the HW you know $|V| \geq 5$. Both graphs have degree sequence $(1, 2, 2, 2, 3)$, are clearly complements, and are nonisomorphic since one has a 3-cycle and the other does not.



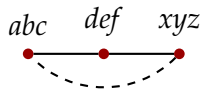
♠♥♣♦

4. Let $X = \{1, 2, 3, 4, 5, 6, 7\}$. Suppose the vertices of G are the 3-element subsets of X , and that two vertices are adjacent if they have empty intersection. Note that G is 4-regular, with $|V_G| = 35$, $|E_G| = 70$. Does G have a 3-cycle? Explain.

Solution

No, three subsets of size 3 from a 7-element set cannot be pairwise disjoint.

If you tried to build a labeling, it would require three vertices labeled with abc, def, xyz , where $\{x, y, z\} \subset \{a, b, c, g\}$, and $\{x, y, z\} \cap \{a, b, c\} = \emptyset$.



5. Prove or disprove: It is possible to decompose K_8 into copies of C_8 .

Solution

K_{2n} will not decompose into cycles (of any size) because the degree of any vertex in K_{2n} is odd, but a decomposition into cycles would require even degree.

Alternately, you could notice that K_8 has $\binom{8}{2} = 28$ elements, which is not divisible by 8. ♠♥♣♦

Circle which you are doing. Then do it.

Homework 2

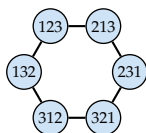
Due: Fri, 3 Feb 2023

Key from text:
 (–) = easier
 (+) = harder
 (!) = useful or
 instructive
 (*) = uses optional
 material

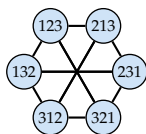
1. **Problem 1.2.17.** (!) Let G_n be the graph whose vertices are the permutations of $(1, \dots, n)$, with two permutations a_1, \dots, a_n , and b_1, \dots, b_n , adjacent if they differ by interchanging a pair of adjacent entries (G_3 shown below). Prove that G is connected.

Solution

Bubble sort, or insertion sort, or any sort that just uses adjacent swaps.



2. **Problem 1.2.20.** Let v be a cut-vertex of a simple graph G . Prove that $\overline{G} - v$ is connected.
3. **Problem 1.2.27.** Let G_n be the graph whose vertices are the permutations of $(1, \dots, n)$ with two permutations (a_1, \dots, a_n) and (b_1, \dots, b_n) adjacent if they differ by switching two entries (G_3 shown below). Prove that G_n is bipartite.



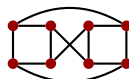
Solution

The easiest way is to know the mathematical fact: every permutation is either even or odd. Then the bipartition is X = the set of even permutations, Y = the set of odd permutations. The hard part would be to prove this fact about parity. Similarly, if you try to show there are no odd cycles, the hard part is essentially to show that parity of permutation is well-defined.



4. **Problem 1.2.28.** (!) In each graph below, find a bipartite subgraph with the maximum number of edges. Prove that this is the maximum, and determine whether this is the only bipartite subgraph with this many edges.



Use the theory, not
brute force.



5. **Problem 1.2.29.** (!) Let G be a connected simple graph not having P_4 or C_3 as an induced subgraph. Prove that G is a biclique (complete bipartite graph).

induced subgraph
defined on p.23

Solution

First, some examples. The graph  has P_4 as a subgraph, but it is not an induced subgraph. The graph  has both P_4 and C_3 as subgraphs, but neither is an induced subgraph. Note that if a simple graph G has C_3 as a subgraph, it is automatically an induced subgraph. Note that if a simple graph G has C_n , $n > 3$, then this cycle has a “chord” (an edge between two elements of the cycle) in G iff it is not induced.

Now, assume G is simple, connected, and has no induced P_4 or C_3 . Then G can have no *induced* C_n , for $n \geq 5$, since that would give an induced P_4 . So any odd n -cycle, $n \geq 5$, has a “chord” connecting two elements of the cycle that splits the cycle into two cycles—one odd, one even. Recursively, get no odd cycles. So we now know G is bipartite. Let the bipartition be X, Y .

To show G is complete, we argue the contrapositive. If G is not complete, fix $x \in X$, $y \in Y$ that are not adjacent. Since G is connected, there is an xy -path. Let P be a minimum length such path. Clearly, it must have length ≥ 3 , and if P is not induced, then it cannot be of minimum length. So P is induced, and must contain an induced P_4 , so we are done. ♠♥♣♦

6. **Problem 1.2.40.** Let P and Q be paths of maximum length in a connected graph G . Prove that P and Q have a common vertex.

Optional bonus question. Let P be the Petersen graph.

- (a) Find a copy of C_{10} in the complement of P .

Solution

For the C_{10} to be in the complement of P , we must have labels that *do* intersect:

$ab, ac, ad, ae, eb, ec, ed, db, dc, cb, ab$

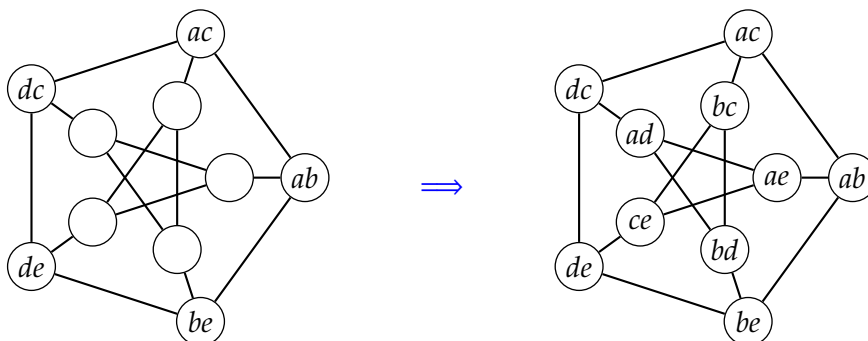
♠♥♣♦

- (b) Is there a copy of P in the complement of P ?

Solution

To ensure that our new P is in the complement of the first, we must have labels that *do* intersect. What worked for me was to label the outer cycle first, but using each element only twice. Probably there is a lemma to explain that. Anyone?

The standard proof of (c) uses eigenvalues, but you can do in an “elementary”, but slightly tedious, way. Use the Kneser labeling.



So, YES, there is a Petersen, call it P' , in the complement of P . Note that there are many edges in the complement of P we did not use, e.g. $(ab) - (bc)$. We are only trying to find a *copy* of P in the complement of the first P , so we are not obligated to add $(ab) - (bc)$ to our graph. The edge $(ab) - (bc)$ is in the complement of $P \cup P'$. ♠♥♣♦

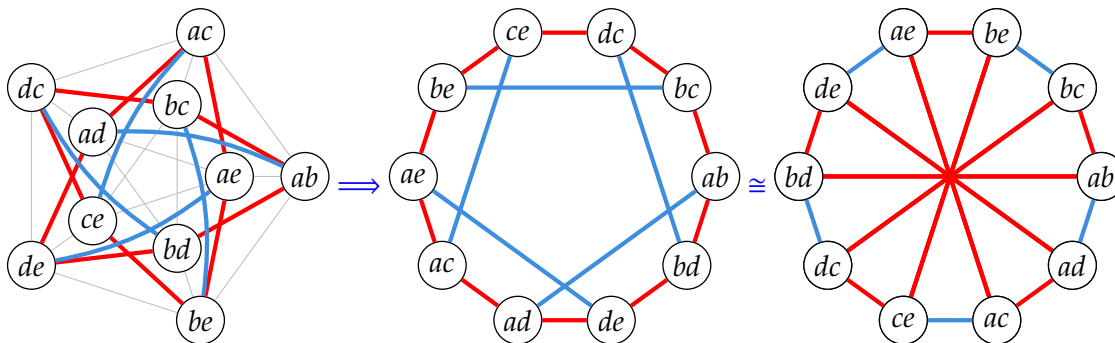
- (c) Are there two edge-disjoint copies of P in the complement of P ?
N.B. Just counting edges tells us there are at most 2 such edge-disjoint copies of P .

Solution.....

Draw the complement of $P \cup P'$ in **red** and **tuftsblue** to complete the decomposition

$$K_{10} = P \cup P' \cup \text{Circ}_{10}(1, 5),$$

not $K_{10} = P \cup P' \cup P''$. (Of course, once you see the red 10-cycle you already know the answer to (c) is NO, assuming you know that P is not Hamiltonian.)



This is not a *complete, general proof* (why?), but we are well on our way. ♠♥♣♦

Homework 3

Due: Fri, 10 Feb 2023

1. **Problem 1.3.8.** (–) Which of the following are graphic sequences? Provide a construction item or a proof of impossibility for each.

(a) (5,5,4,3,2,2,1),

(b) (5,5,4,4,2,2,1,1),

(c) (5,5,5,3,2,2,1, 1),

(d) (5,5,5,4,2,1,1,1).

Solution

You can use the Havel-Hakimi algorithm to actually build the graph, if the sequence is indeed graphic. But nobody showed this on their HW, so I will ask you to redo part (a) on HW 5, and to show the steps for building the graph.

Hence, this problem won't be on Q3. ♠

2. **Problem 1.3.12.** (!) Prove that an even graph has no cut-edge. For each $k \geq 1$, construct a $(2k + 1)$ -regular simple graph having a cut-edge.

The solution manual gives 3 proofs: (contradiction), (construction/extremality), (prior results), and 3 different constructions. See if you can generate more than one of these. But one is sufficient!

3. **Problem 1.3.17.** (!) Let G be a graph with at least two vertices. Prove or disprove:

(a) Deleting a vertex of degree $\Delta(G)$ cannot increase the average degree.

(b) Deleting a vertex of degree $\delta(G)$ cannot reduce the average degree.

4. **Problem 1.3.20.** (!) Count the cycles of length n in K_n , and the cycles of length $2n$ in $K_{n,n}$.

Easy to get wrong if you are not careful.

Solution

(a) Fix an arbitrary vertex in the first position of the cycle. Then every cycle can be gotten by $(n - 1)$ choices for the remaining vertices. But every cycle is now counted twice since the reverse order gives the same cycle, so the answer = $(n - 1)!/2$.

(b) As above, fix an arbitrary vertex v in the first position of the cycle. Then we get every cycle with the following choices, assuming v is in the red half of the bipartition:

$$(n)(n - 1)(n - 1)(n - 2)(n - 2) \cdots (2)(1)(1)$$

As above, each cycle is counted twice, so the answer = $n!(n - 1)!/2$



5. **Problem 1.3.36.** Let G be a 4-vertex graph whose list of subgraphs obtained by deleting one vertex appears below. Determine G .



6. **Problem 1.3.37.** Let H be a graph formed by deleting a vertex from a loopless regular graph G with $n(G) \geq 3$. Describe (and justify) a method for obtaining G from H .

Solution

Note that this is explicitly for non-simple graphs. Suppose the reconstruction deck for regular loopless G is



Then $|E| = 4 \cdot 4 / (4 - 2) = 8$, so $4d = 2 \cdot 8, d = 4$, so G is



Bonus problems:

7. **Problem 1.3.25. (!)** (promoted/demoted to bonus question) Prove that every cycle of length $2r$ in a hypercube is contained in a subcube of dimension at most r . Can a cycle of length $2r$ be contained in a subcube of dimension less than r ?
8. **Problem 1.3.26. (!)** Count the 6-cycles in Q_3 . Prove that every 6-cycle in Q_k lies in exactly one 3-dimensional subcube. Use this to count the 6-cycles in Q_k for $k \geq 3$.
9. **Problem 1.2.36. (+)** Alternative characterization of Eulerian graphs.
- Prove that if G is Eulerian and $G' = G - uv$, then G' has an odd number of u, u -trails that visit v only at the end. Prove also that the number of the trails in this list that are not paths is even. (Toida [1973])
 - Let u be a vertex of odd degree in a graph. For each edge e incident to v , let $c(e)$ be the number of cycles containing e . Use $\sum_e c(e)$ to prove that $c(e)$ is even for some e incident to v . (McKee [1984])
 - Use part (a) and part (b) to conclude that a nontrivial connected graph is Eulerian if and only if every edge belongs to an odd number of cycles.

10. **Problem 1.3.16.** (+) For $k \geq 2$ and $g \geq 2$, prove that there exists a k -regular graph with girth g . (Hint: To construct such a graph inductively, make use of a $(k-1)$ -regular graph H with girth g and a graph with girth $\lceil g/2 \rceil$ that is $n(H)$ -regular. (Comment: Such a graph with minimum order is a (k, g) -cage.) (Erdős-Sachs [1963])
See [https://en.wikipedia.org/wiki/Cage_\(graph_theory\)](https://en.wikipedia.org/wiki/Cage_(graph_theory)).

Quiz 2

Wed, 8 Feb 2023

Hint: There are two to “prove” and two to “disprove”.

1. Prove or disprove (circle one, then do it.): Let G be simple. If G is not connected, then \overline{G} must be connected.

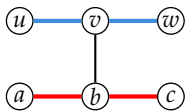
Solution

Suppose G is not connected, with components G_1, \dots, G_k . Then every vertex in G_i is adjacent in \overline{G} to every vertex in G_j , if $i \neq j$. So every pair of vertices is connected by a path of length at most 2. \diamond

2. Prove or disprove (circle one, then do it.): Two maximal paths P, Q in a connected simple graph G must share a vertex.

Solution

Counterexample: the paths (u, v, w) and (a, b, c) are both maximal, but not of maximum length.



\diamond

3. Prove or disprove (circle one, then do it.): Let G_n be the graph whose vertices are the permutations of $\{1, \dots, n\}$, with two vertices $a = (a_1, \dots, a_n)$, $b = (b_1, \dots, b_n)$ adjacent if b can be gotten from a by swapping a_1 with some a_j , $j > 1$. Then G connected.

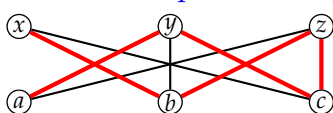
Solution

G is connected since any number $j > 1$ can be put in the j th place with at most 2 swaps. Hence, any vertex is connected to $(1, 2, \dots, n)$ by a path of length $< 2n$. \diamond

4. Prove or disprove (circle one, then do it.): Suppose G is connected bipartite with bipartition A, X . Let $a \in A$, $x \in X$ such that a, x are not adjacent, but they are connected via the ax -path P . Then P contains an induced P_4 .

Solution

Counterexample: the red path (a, y, c, z, b, x) does not contain an induced P_4 .



\diamond

Homework 4

Due: Fri, 17 Feb 2023

- Take a look, but do not turn in:

1.3.38 Let G be a graph with at least 3 vertices. Prove that G is connected if and only if at least two of the subgraphs obtained by deleting one vertex of G are connected. (Hint: Use Proposition 1.2.29.)

This problem is potentially confusing: give an incorrect interpretation that could easily be made under exam stress. Then note: (\Rightarrow) is easy using the hint, and (\Leftarrow) is not hard, but has a subtlety (a special case easy to miss).

- Problem 1.3.40.** (!) Let G be an n -vertex simple graph, where $n \geq 2$. Determine the maximum possible number of edges in G under each of the following conditions.

- G has an independent set of size a .

Solution

Clearly, we must have $\binom{n}{2} - \binom{a}{2}$ edges. Let $G = K_n$ and let $H \cong K_a$ be a subgraph of G . Let M be the subgraph of G with $V_M = V_G$, $E_M = E_G - E_H$. Then V_H is an indep set of size a in M , and M has the desired size. ♠

- G has exactly k components.

The issue here is whether you should have few large K_i s or many smaller K_i s.

- skip** G is disconnected.

- Problem 1.3.46.** Prove or disprove: Whenever the algorithm of Theorem 1.3.19 is applied to a bipartite graph, it finds the bipartite subgraph with the most edges (the full graph).

We did Theorem 1.3.19 in class, but there was some confusion about why/how you could switch a vertex v from one side of the bipartition to the other in order to increase $\deg(v)$. The algorithm does this operation until it achieves $e(H) \geq e(G)/2$ (where H is the bipartite subgraph we are building).

At any rate, make sure you understand this algorithm (future-quiz threat ☺).

- In class I gave the characterization of trees: Let T be a graph with n vertices. Then the following statements are equivalent.

- T is a tree.
- T contains no cycles and has $n - 1$ edges.
- T is connected and has $n - 1$ edges.
- T is connected, and every edge is a cut-edge.

5. Any two vertices of T are connected by exactly one path.
6. T contains no cycles, and for any new edge e , the graph $T + e$ has exactly one cycle.

Prove (5) \Rightarrow (6).

Solution

T contains no cycles since any two vertices on a cycle have two different paths between them, consisting of the opposite routes around the cycle.

Furthermore, the addition of any new edge e to T will create a cycle, since the endpoints of e , say u and v , are already connected by a path in T .

To show that this cycle is unique, suppose two cycles were created. They both would contain edge e , and the long way around each of these cycles would then be two different uv -paths in T , contradicting the assumption. ♠

4. **Problem 2.1.4.** (–) Prove or disprove: Every graph with fewer edges than vertices has a component that is a tree.

Solution

If $|E_G| < |V_G|$ then G has a component H with $|E_H| < |V_H|$. $|E_H| = n - 1$ we know H is a tree and we are done.

Now, for a contradiction, assume $|E_H| = k < n - 1$. Then add $(n - 1) - k$ new edges to H to get H' . Then $|E_{H'}| = n - 1$ (and H' is still connected), so H' is a tree, and each edge is a cut edge. So when we remove the new edges to get back to H we have a disconnected graph, contradiction. So H has $n - 1$ edges, and must be a tree. ♠

5. **Problem 2.1.18.** (!) Prove that every tree with maximum degree $\Delta > 1$ has at least Δ vertices of degree 1. Show that this is best possible by constructing an n -vertex tree with exactly Δ leaves, for each choice of n, Δ , with $n > \Delta \geq 2$.

Solution

The following has an error. Can you find it?

Let $\deg(v) = \Delta$. Then for each $u \in N(v)$ there is a maximal subtree T_u not containing v , and these are pairwise disjoint. Inductively, each contains at least one leaf, and any leaf of T_u is a leaf of T , so T has at least Δ leaves. ♠

Bonus problems

6. **1.3.52.** Prove that every n -vertex triangle-free simple graph with the maximum number of edges is isomorphic to $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$. (Hint: Strengthen the proof of Theorem 1.3.23.)

We talked about 1.3.23 in class, so you would need to understand that proof in order to follow the hint.

7. Code up the algorithm for Prüfer encoding and decoding. You can use whatever data structure you want to represent trees: edge list, adjacency matrix, ...

Quiz 3

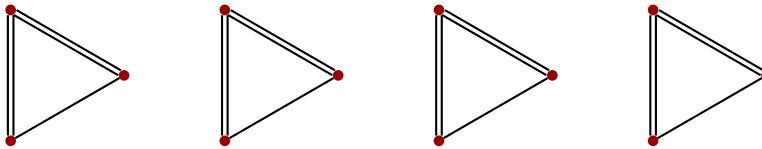
Wed, 15 Feb 2023

1. Let G be gotten by adding a new vertex v to K_n , and making v be adjacent to exactly one vertex $u \in K_n$ (so that $\deg(v) = 1$). Count the paths of maximal length in G .

Solution

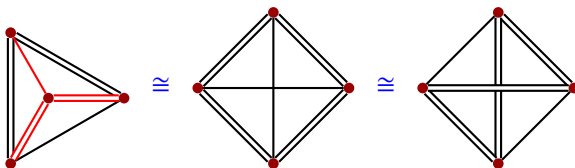
Clearly, a maximal path in G must go through u . Every cycle in K_n gives two maximal paths (in K_n) through ending in u , and each can be extended in a unique way to a maximal path in G that includes v . So the number is $(n-1)!$. ♥

2. Suppose that G is a regular loopless graph with the following reconstruction deck. Find G .



Solution

$|E| = 5 \cdot 4 / (4 - 2) = 10$, so $4d = 2|E|$, $d = 5$. So G is



Note: Almost everyone got this correct, but showed no work. You should have computed the degree first. Reconstructing the graph is trivial once you have that. ♥

Homework 5

Due: Fri, 24 Feb 2023

- Problem 1.3.8.** Redo part (a) of this one from HW 3, but generate the graph using the intermediate steps from the Havel-Hakimi algorithm. Of course, show your steps.
- Problem 2.1.49.** Let G be a simple graph. Prove that $\text{rad}(G) \geq 3 \Rightarrow \text{rad}(\overline{G}) \leq 2$.
- Problem 2.2.1.** (–) Determine which trees have Prüfer codes that
 - contain only one value,
 - contain exactly two values, or
 - have distinct values in all positions.
- Problem 2.2.24.** Of the n^{n-2} trees graphs with vertex set $\{0, \dots, n-1\}$ that have $n-1$ edges, how many are gracefully labeled by their vertex names?
- Problem 2.3.29.** (–) The game of Scrabble has 100 tiles as listed below. This does not agree with English; "S" is less frequent here, for example, to improve the game. Pretend that these are the relative frequencies in English, and compute a prefix-free code of minimum expected length for transmitting messages. Give the answer by listing the relative frequency for each length of codeword. Compute the expected length of the code (per text character). (Comment: ASCII coding uses five bits per letter; this code will beat that. Of course, ASCII suffers the handicap of including codes for punctuation.)

Correction as suggested by West in the Soln. Manual.

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	\emptyset
9	2	2	4	12	2	3	2	9	1	1	4	2	6	8	2	1	6	4	6	4	2	2	1	2	1	2

Bonus Problems

- Problem 2.2.23.** (!) Prove that if the Graceful Tree Conjecture is true and T is a tree with m edges, then K_{2m} decomposes into $2m-1$ copies of T . (Hint: Apply the cyclically invariant decomposition of K_{2m-1} for trees with $m-1$ edges from the proof of Theorem 2.2.16.)
Note: We did not do Theorem 2.2.16 in class, so you have to read and understand that one first.
- Problem 2.3.30.** Consider n messages occurring with probabilities p_1, \dots, p_n , such that each p_i is a power of $1/2$ (each $p_i \geq 0$ and $\sum p_i = 1$).
 - Prove that the two least likely messages have equal probability.
 - Prove that the expected message length of the Huffman code for this distribution is $-\sum p_i \lg p_i$.

Practice Quiz on HW 4/5

Thu, 23 Feb 2023

From HW 4

1. A tree is defined as a connected acyclic graph. Prove that a tree with n vertices has exactly $n - 1$ edges. State explicitly any lemma you need to use (but you don't need to prove it).

I will not typically ask you to simply reproduce a proof we have done, but this one is pretty basic and important, so ...

Solution

Proof by induction (on n).

base case: $n = 1, 2$ easy.

induction step: Suppose T has $n + 1$ vertices.

Then by the lemma below, T has a leaf v .

Let $T' = T - v$.

Since T' is still connected (and acyclic), T' is a tree.

Clearly, $|V(T')| = n$, and since $\deg_T(v) = 1$, we have $|E(T')| = |E(T)| - 1$.

Inductively, $|E(T')| = n - 1$, so $|E(T)| = n$, as desired.

Need: **Lemma**. Any tree with at least one edge has at least two leaves. 

2. Let G be an n -vertex simple graph, where $n \geq 2$, and fix positive integers $a > k$. Determine the maximum possible number of edges in G if G has an independent set of size a and k components.

Solution

Note that $[k]$ viewed as a graph is just k isolated vertices.

$G = (K_{n-k+1-a} \vee [a]) + [k - 1]$.



3. Prove or disprove: Suppose G is bipartite with bipartition A, B , and suppose $b \in B$. Let $A' = A \cup \{b\}$, $B' = B - \{b\}$, so this "incorrect" bipartition is off by only one vertex.

If we apply the algorithm of Theorem 1.3.19 to G with the initial bipartition A', B' , then it finds the bipartite subgraph with the most edges (the full graph).

4. Draw the tree with Prüfer sequence $(1, 2, 3, 4, 5)$.

From HW 5

5. Give a family of graphs G_n with $\text{rad}(G_n) < \text{diam}(G_n)$, showing that the average eccentricity can be arbitrarily close to the radius (the min eccentricity).

Solution

C_{2n} plus one new vertex adjacent to exactly one vertex of C_{2n} .

Alternatively: $K_n - e$.



6. Give a family of graphs G_n with $\text{rad}(G_n) < \text{diam}(G_n)$, showing that the average eccentricity can be arbitrarily close to the diameter (the max eccentricity).

Solution

The star graphs S_n .



7. Suppose we are given the symbol probabilities: $p(a) = 0.2$, $p(b) = 0.05$, $p(c) = 0.1$, $p(d) = 0.1$, $p(e) = 0.25$, $p(f) = 0.15$, $p(g) = 0.15$. If we build a Huffman code using the algorithm from class, do we always get the same codeword lengths, even if we make different choices to break ties (for equal minimal probabilities) at each step?

8. Prove or disprove: If G is a simple graph with $\text{rad}(G) \geq 4$, then $\text{rad}(\overline{G}) \leq 3$.

Exam 1 Hints

The exam is on **Mon, 27 Feb 2023** in class. No notes or books, but I will supply any definitions you ask for. When building the exam, I will be looking at

1. quizzes
2. homework
3. problems in the text marked with (-), i.e., the easy ones

problems from quizzes might show up on the exam without modification,
problems from the assigned hw will typically not (usually too hard without modification).

Exam 1

Mon, 27 Feb 2023

Name on every leaf. Please print your name 3 times (once on each leaf).


Use the answer boxes.

Show your work.

All problems worth 10 points. All parts of a problem are of equal value.

Work exactly 10 of the problems.

On the two problems that you don't work place a **large visible** star \star . You can only assign a \star to an entire problem, not just a part. You **must** assign two \star 's.

1. A difficult problem ...

2. An easy problem ...

Good luck!

1. (a) Both of the following graphs decompose into a C_7 and P_7 . Determine if they are isomorphic. Explain.



Solution

Not isomorphic. One complement has a 3-cycle, the other doesn't.



- (b) Determine if the graphs below are isomorphic. Explain.

Note: It's not obvious since they have the same degree sequence, they both have 3-, 4-, 5-, 6, and (at least two) 7-cycles, and the complements don't really seem to help here.

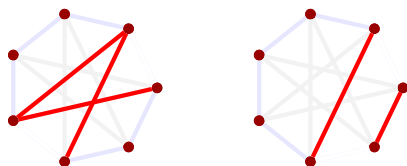
Hint: Consider the subgraphs induced by the vertices of a particular degree.



Solution

Note that the complements don't make the answer obvious.

In fact, these graphs are NOT isomorphic. In both graphs there are 4 vertices of degree 3. Consider the subgraph induced by these degree-3 vertices. In the first graph this is a P_4 , while in the second it is $P_2 + P_2$. Alternatively, you could look at the subgraphs induced by the degree-4 vertices: you get a 3-cycle and a P_3 .



2. Prove: If G is simple, connected, r -regular, with $n = |V|$ odd, then G has an Eulerian circuit.

Solution

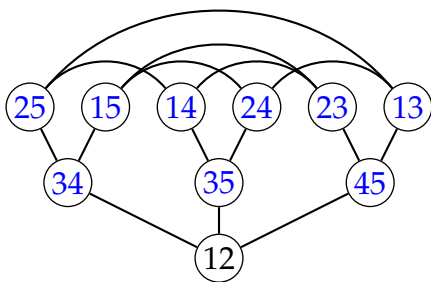
$$\left. \begin{array}{l} \sum \deg(v_i) = 2|E| \\ \sum \deg(v_i) = rn \end{array} \right\} \Rightarrow r \text{ even.}$$

Hence, G is a connected even graph, so it has an Eulerian circuit.



3. The graph below is 3-regular with 10 vertices. Is it the Petersen graph?
Hint: Kneser labeling. No need to show any work beyond the labeling if it succeeds. If it fails, explain *where/how* it fails.

YES / NO (circle one)



Solution

YES, it is Petersen since the labeling is possible.



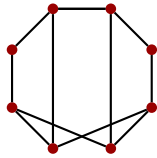
4. Prove: If G is simple and not connected, then \overline{G} must be connected.

Solution

Suppose G is not connected, with components G_1, \dots, G_k . Then every vertex in G_i is adjacent in \overline{G} to every vertex in G_j , if $i \neq j$. So every pair of vertices is connected by a path of length at most 2.

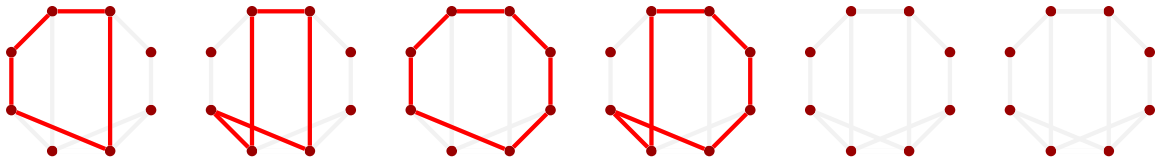


5. Let $G =$



- (a) The number of odd cycles in $G =$ 8 (just brute force it).

They come in vertically symmetric pairs. Draw **ONE** of each such pair. *There are more templates below than you need.*



Solution

FWIW: This graph is isomorphic to the one from the HW:

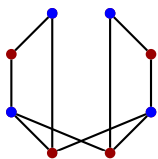
The 8 odd cycles are those above and their reflections about the vertical.



- (b) The number of edges in the largest bipartite subgraph of $G =$ 10.

Solution

All the odd cycles share the top central edge, so we only need to remove that one edge. Hence, the largest bipartite subgraph of G has 10 edges.



- (c) Is it possible for the algorithm of Theorem 1.3.19* to give this largest subgraph if it

**Start with any partition X_0, X_1 of $V(G)$.*

If $v \in X_i$ is adjacent to more vertices in X_{1-i} than to vertices in

starts on an “incorrect” bipartition? Explain.

Hint: Consider the next-to-last step of this process, i.e., the last bipartition before moving a vertex to get the correct partition.

YES / NO (circle one)

Solution

No. It is easy to see that moving any one vertex from one side of the correct partition to the other still gives a bipartite graph that has at least 5 edges, so the algorithm would terminate at this next-to-last bipartition. ♡

6. (a) Determine if the degree sequence $(5, 5, 5, 4, 2, 1, 1, 1)$ is graphic (and show a graph if it is).

YES / NO (circle one)

Solution

$55542111 \rightarrow 4431011 = 4431110 \rightarrow 320010 = 321000 \rightarrow 10(-1)00$ is NOT graphic. ♡

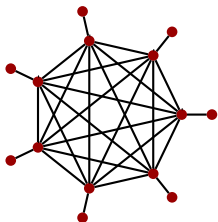
- (b) Prove: For a simple graph G , two vertices must have the same degree.

Hint: Keep in mind the possible degrees for the n vertices.

Solution

The possible degrees are $\{0, 1, \dots, n-1\}$. So the only degree sequence (of length n) has all these degrees. But then $\deg(v_n) = n-1$ and $\deg(v_1) = 0$ are not compatible. ♡

7. Suppose $V(K_n) = \{v_1, \dots, v_n\}$ and let $U_n = \{u_1, \dots, u_n\}$ be new vertices. Define the graph $G_n = K_n + U_n$ + the edges $\{u_i v_i : 1 \leq i \leq n\}$. G_7 is pictured:



- (a) The number of *maximal length* paths of $G_n = \boxed{\frac{n!}{2}}$.

Solution

The maximal length paths in K_n and G_n are in bijection, so there are $\frac{n!}{2}$ such paths.

N.B. In K_n a maximal path is a maximal length path. But in G_n there are maximal paths of length $k+2$, for all $k = 1, \dots, n$. ♡

(b) The number of *maximal* paths of *minimal length* in $G_n =$

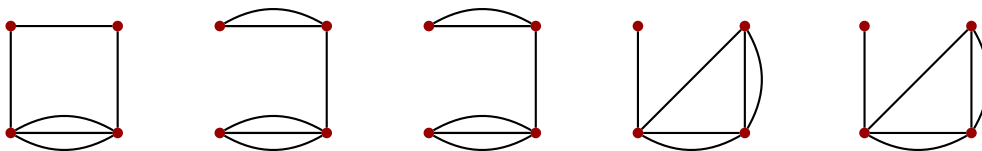
$$\boxed{\binom{n}{2}}$$

Solution

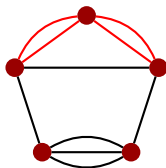
They are length 3, and there are $\binom{n}{2}$ such paths.

♥

8. Suppose G is a regular loopless graph with the following reconstruction deck. Find G .



Draw your reconstruction using these nodes:



Solution

Clearly, $|V| = 5$. Each edge appears in $n - 2$ cards in the deck, so $|E| = 5 \cdot 6 / (5 - 2) = 10$. Then by the Handshaking Lemma, $\sum \deg(v) = 2 \cdot 10$.

But also, $\sum \deg(v) = 5r$, so $5r = 20$, and hence, $\boxed{r = 4}$.

Then G can be reconstructed from any card in the deck by adding edges to the missing vertex to ensure 4-regularity.

♥

9. Let T be a tree with n vertices, with average degree a . Solve for n in terms of a .

$$n = \frac{2}{2-a}$$

Solution

$$a = \frac{\sum_v \deg v}{n} = \frac{2|E|}{n} = \frac{2(n-1)}{n}. \text{ Solve for } n: n = \frac{2}{2-a}.$$

♥

⇐ No work
required.
Just give
the formula.

10. (a) Among all labeled graphs on n vertices, the fraction that are trees =

$$\frac{n^{n-2}}{2^{\binom{n}{2}}}.$$

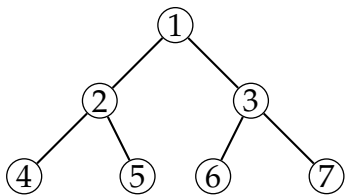
Solution

$$\frac{\text{number of labeled trees}}{\text{number of labeled graphs}} = \frac{n^{n-2}}{2^{\binom{n}{2}}}$$



- (b) The Prüfer sequence for the full binary tree shown is

$\langle 2, 2, 1, 3, 3 \rangle$.

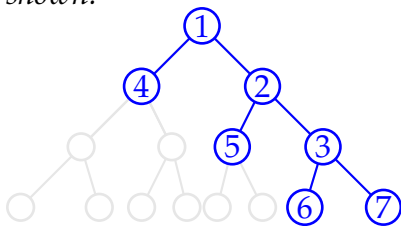


Solution

remove 4, write down 2
remove 5, write down 2
remove 2, write down 1
remove 1, write down 3
remove 6, write down 3



- (c) Draw the tree for the Prüfer sequence $\langle 1, 2, 2, 3, 3 \rangle$ as a subgraph of the full binary tree shown.



Solution

$L = 1234567$

$P = 12233$

add edge $\overline{14}$, remove 4 from L , remove 1 from P

$L = 123567$

$P = 2233$

add edge $\overline{21}$

$L = 23567$
 $P = 233$
 add edge $\overline{25}$

 $L = 2367$
 $P = 33$
 add edge $\overline{32}$

 $L = 367$
 $P = 3$
 add edge $\overline{36}$

 $L = 37$
 $P =$
 add edge $\overline{37}$



11. Recall: In a connected graph G ,

- $d(u, v)$ = the minimum length of a uv -path,
- $\epsilon(v) = \max_u \{d(v, u)\}$,
- $\text{diam}(G) = \max_v \{\epsilon(v)\}$
- $\text{rad}(G) = \min_v \{\epsilon(v)\}$
- $\text{center}(G) = G[\{v : \epsilon(v) = \text{rad}(G)\}]$ = subgraph induced by vertices with min $\epsilon(v)$.

Note: Let $V(P_{2n}) = \{v_1, \dots, v_{2n}\}$, so that $|V| = 2n$, $|V| \neq n$.

(a) $\text{diam}(P_{2n}) = \boxed{2n - 1}$

*For work,
just draw a
picture.*

(b) $\text{rad}(P_{2n}) = \boxed{n}$

(c) $\text{center}(P_{2n})$ is induced by $\boxed{\{v_n, v_{n+1}\}}$

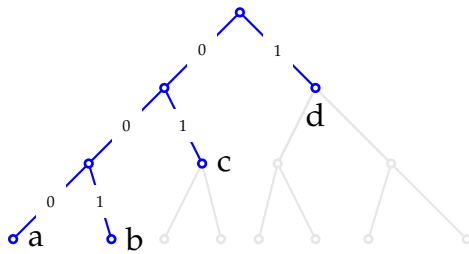
Solution

$\text{center}(P_{2n}) \cong P_2$, with vertices v_n, v_{n+1}



12. Suppose symbols $\{a, b, c, d\}$ have probabilities $p(a) = p(b) = 1/8$, $p(c) = 1/4$, $p(d) = 1/2$.

(a) Draw the Huffman tree as a subtree of the binary tree below.



Solution

merge a and b . merge (a, b) and c . merge $((a, b), c)$ and d .



- (b) The expected length (in bits) of a message with 100 symbols = 175.

Note: If we used a full binary tree rather than Huffman, this message would be exactly 200 bits long.

Solution

expected codeword length = $3 \cdot \frac{1}{8} + 3 \cdot \frac{1}{8} + 2 \cdot \frac{1}{4} + 1 \cdot \frac{1}{2} = 1.75$, so expected message length = 175.



Midsem Student Reviews

Wed, 1 Mar 2023

1. Is there something the instructor should KEEP about the course to support your learning this semester? Explain briefly.
2. Is there something the instructor should QUIT doing in the course to better support your learning this semester? Explain briefly.

Student responses

maybe one hw problem not from the textbook, ... an easier supplemental problem to get at the basics

Perhaps. You can always take a look at a couple of the "easy" problems in the text, those marked with (-), for quiz / exam prep, etc. And come to OH for discussion of the basics.

quizzes don't help understand ... either too close to hw or too different

Hopefully it gets cleared up when you see the solutions(?)

inconsistent notation V_G vs $V(G)$

"In the wild" you will see many variations, so my idea is to use these mildly different notations often enough that you become indifferent to them.

lecture gets slowed down by entertaining too many questions

Yes, but a difficult call. Different students will prefer different approaches. If I can detect a consensus I will try to follow that.

too long on a theorem if it isn't essential

don't just say "trivial" or "obvious"

Acknowledged, but a difficult call, i.e., it's difficult to know what the audience needs. Absolutely no problem if you just ask for clarification if it's not really "obvious" (and you won't be the only one).

sometimes proofs in lecture are incomplete

Just for the sake of saving time. Please ask after class, or in OH, for more discussion/details if you want.

stop using maximal length, use maximum length

Same as "notation" comment above. You WILL see both, and you need to know when "maximal" and "maximum" are the same or different. (For "length" they are the same because \mathbb{R} is well-ordered.)

3. Is there something the instructor could START doing in the course to better support your learning this semester? Explain briefly.

Student responses

collaborative, group work

Nice idea, but not sure if we can implement it here. We can discuss in class if you bring it up.

more counterexamples, explore how theorems break down (when hypotheses are relaxed)

Great idea. Ideally you should do this with every theorem, in order to get full understanding. For us, it's always just a question of time. . .

review more proof techniques

Mostly the standard proof techniques you would learn in cs61. The rest are typically bespoke ideas particular to graph theory, or a particular problem in graph theory. All we can do is learn them as they come up.

sometimes students raise their hands or want to participate and aren't noticed

♡ ♡ ♡

This is the number one question I want to respond to. Please don't let this happen! Just voice out, or if you see someone is not getting noticed, voice out on their behalf. Lecture is informal and you are encouraged to interrupt and participate.

♡ ♡ ♡

either more lecture content (answer fewer questions), or easier hw

slides or notes from lecture

Maybe Adam can take photos of the boards and post?

one easy question at start of quizzes

Ok, good idea ☺

color code boardwork, organize with a clear direction for the day's lecture

give intuition for theorems not proved

Ok. also, just ask in class, or in OH.

other resources (texts)?

Other texts I have taught this class from:

(a) Graph Theory and Its Applications 3e, by Gross, Yellen, Anderson (the 2e is even better in some ways)

(b) Introduction to Graph Theory 5e, by Wilson

pictures of the board from class

Maybe Adam can take photos and post?

4. Do you have any other comments, concerns, or anything else to add?

Student responses

... the potential of required attendance at Friday OH, but I have a class ...

This comment was in jest—I meant you really should come to OH, and would *like* to require it. I don't have the right to demand any specific time of you outside of scheduled class (nor do any of your other professors).

text not beginner friendly

Yes. I was following the previous two faculty here who have taught this class. I might choose Gross & Yellen, or Wilson, if I had to choose again.

break quizzes into an easier in-class part, then harder collaborative take-home

Interesting idea. Not sure we can do this, but interesting ...

5. Compared to other similar courses (e.g., 3-credit 150-level CS courses), is this course **easier**, **harder**, or about the **same**? (Circle one) Explain briefly.
6. Compared to other similar courses (e.g., 3-credit 150-level CS courses), do you feel you are learning **more**, **less**, or about the **same**? (Circle one) Explain briefly.
7. Please rate how each element of the class contributes to your learning

	Low contribution	moderate contribution	strong contribution
homework			
textbook			
exams			
lectures			

Homework 6

Due: Fri, 10 Mar 2023

1. **Problem 5.1.31.** (!) Prove that a graph G is m -colorable if and only if $\alpha(G \square K_m) \geq n(G)$. (Berge [1973, 379-80])

Solution.....

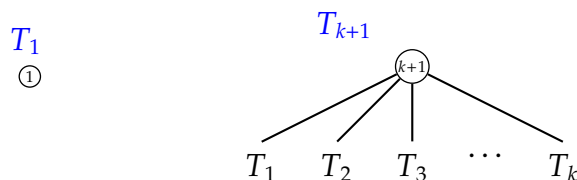
(\Rightarrow) Fix an m -coloring $c : V_G \rightarrow [m]$. For a fixed color i , define $X_i \subseteq V(G \square K_m)$ to be $X_i = \{(v, i) : c(v) = i\}$. Note that X_i is an independent set, and that X_i, X_j are disjoint if $i \neq j$. Now define $X = \cup_i X_i$. Then X is an independent set, and $|X| = n$.

(\Leftarrow) Assume X is an independent set in $G \square K_m$, and $|X| = n$. Note that X cannot contain two vertices $(g, i), (g, j)$ that use the same element of G , so all elements of G get used in some vertex of X . Define $G_i = \{g : (g, i) \in X\}$, for $1 \leq i \leq m$, and note that G_i is an independent set in G . Hence, the function $c(g) = i$, if $g \in G_i$, is an m -coloring of G . ♠

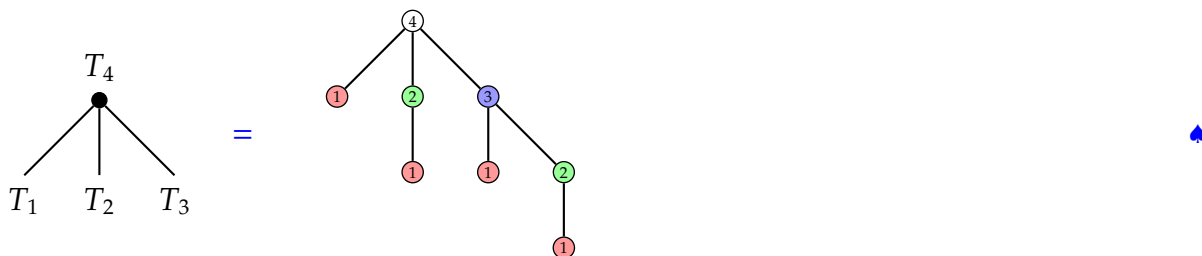
2. **Problem 5.1.34.** (!) For all $k \in \mathbb{N}$, construct a tree T_k with maximum degree k and an ordering σ of $V(T_k)$ such that greedy coloring relative to the ordering σ uses $k+1$ colors. (Hint: Use induction and construct the tree and ordering simultaneously. Comment: This result shows that the performance ratio of greedy coloring to optimal coloring can be as bad as $(\Delta(G) + 1)/2$. (Bean [1976])

Solution.....

In order to force the color $k+1$ on a node, it must have k neighbors using all the colors $1, \dots, k$. For each color i , recursively define a tree-and-ordering T_i that requires the root node to be colored i . T_1 is just a single node labeled "1", and (recursively) T_{k+1} is:



The ordering is given by the labels in the recursively defined trees, e.g.,



3. **Problem 5.2.7.** (!) Given an optimal coloring of a k -chromatic graph, prove that for each color i there is a vertex with color i that is adjacent to vertices of the other $k - 1$ colors.
4. **Problem 5.2.15.** (!) Prove that a triangle-free graph with n vertices is colorable with $2\sqrt{n}$ colors. (Comment: Thus every k -chromatic triangle-free graph has at least $k^2/4$ vertices.)

Solution.....

This is the same solution as in the text's answer book. I just tried to present it in a way that might indicate what steps might have lead you to the theorem. It is easy to understand the solution, but still feels like it would be hard to guess ahead of time that the theorem might be true, so that then you could go attempt to prove it.

Recall: Every graph can be $(\Delta_G + 1)$ -colored. The idea below is to iteratively remove nodes until $\Delta_{G'} + 1 \leq k$, where $k = \text{target number of colors}$.

At each iteration we will use 1 new color to color $(\Delta_G + 1)$ nodes, then remove them.

Let $G_0 = G$. Then for $i \geq 0$:

Let $v_i \in G_i$ s.t. $\deg(v_i) = \Delta_i \stackrel{\text{def}}{=} \Delta_{G_i}$

Color v_i with c_0

Color all $u \in N_{G_i}(v_i)$ with c_i

Define $G_{i+1} = G_i - (\{v_i\} \cup N_{G_i}(v_i))$

After t -many iterations (at stage $t - 1$) we have used $(1 + t)$ -many colors, and need only Δ_t more to color the rest. (We don't need $\Delta_t + 1$ because color c_0 can be use on G_t .) So the total number of colors we need is $t + \Delta_t + 1$. In order to minimize this quantity, we stop when $t \geq \Delta_t$. Let this critical t -value be $t^* = \min\{t : t \geq \Delta_t\}$. Then $1 + 2t^*$ -many colors will suffice.

Note: We might have $\Delta_G \ll 2\sqrt{n}$, so this theorem could give a very suboptimal bound. But what if we don't know Δ_G ? The point of this hw/theorem is to give a bound for χ_G in terms of only n (when we don't have any knowledge about Δ_G).

The clever part is realizing you can express t^* in terms of n .

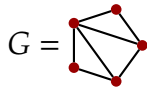
At stage $t^* - 1$, the number of uncolored nodes is $|G_{t^*}| = n - (1 + \Delta_0) - (1 + \Delta_1) - \dots - (1 + \Delta_{t^*-1})$. Since $\Delta_0 \geq \dots \geq \Delta_{t^*-1}$, we clearly have $0 \leq |G_{t^*}| \leq n - t^*(1 + \Delta_{t^*-1})$. But, by definition of t^* we have $t^* - 1 < \Delta_{t^*-1}$. Substituting this gives $0 < n - t^*(1 + t^* - 1) = n - (t^*)^2$. Hence, $t^* < \lceil \sqrt{n} \rceil$, so $1 + 2t^* \leq 2\lceil \sqrt{n} \rceil$, and $2\sqrt{n}$ colors suffice. ♠

5. **Problem 5.3.4.** Notation: $\chi(G; k) = \chi_G(k)$

(a) Prove that $\chi(C_n; k) = (k - 1)^n + (-1)^n(k - 1)$.

- (b) For $H = G \vee K_1$, prove that $\chi(H; k) = k \cdot \chi(G; k-1)$, i.e., $\chi_H(k) = k \cdot \chi_G(k-1)$, or even better, $P_H(k) = kP_G(k-1)$. From this and part (a), find the chromatic polynomial of the wheel $C_n \vee K_1$.

6. Verify the graph G has chromatic polynomial $\chi_G(k) = k^5 - 7k^4 + 18k^3 - 20k^2 + 8k$.



Solution.....

$$\begin{aligned}
 \text{Graph } G &= \text{Graph 1} - \text{Graph 2} \\
 &= \left(\text{Graph 3} - \text{Graph 4} \right) - \left(\text{Graph 5} - \text{Graph 6} \right) \\
 &= \left(k(k-1)^3(k-2) - k(k-1)^2(k-2) \right) - \left(((k-1)^4 + (k-1)) - k(k-1)^2 \right)
 \end{aligned}$$

then after a half-page of tedium,

$$= \boxed{k^5 - 7k^4 + 18k^3 - 20k^2 + 8k}.$$

Here's maybe a nicer (tedium-free) way to get the factored version of $\chi_G(k)$.

$$\begin{aligned}
 \text{Graph } G &= \text{Graph 1} - \text{Graph 2} \\
 &= \left(\text{Graph 3} - \text{Graph 4} \right) - \text{Graph 5} \\
 &= (k-2) \cdot \text{Graph 6} \\
 &= (k-2) \left(\text{Graph 7} - \text{Graph 8} \right) \\
 &= (k-2) \left(\left(\text{Graph 9} - \text{Graph 10} \right) - \text{Graph 11} \right) \\
 &= (k-2) \left((k-2) \cdot \text{Graph 12} \right) \\
 &= \boxed{(k-2)^3(k-1)k}
 \end{aligned}$$



Problem 5.3.12. (+) Coefficients of $\chi(G; k)$.

Skip this one, but recall that the leading term of $\chi_G(k)$ is k^n , so the coefficient is 1, and that the second term is $-|E_G|k^{n-1}$ (proved in your text). Compare these to part (a) below. Also, there was a question in class about roots of $\chi_G(k)$. See this random paper: https://www.cs.tufts.edu/comp/150GT/documents/Algebraic_properties_of_chromatic_roots.pdf for some info. I was struck by Theorem 2: the location of chromatic roots in \mathbb{C} is related to the theory of phase transitions. (!)

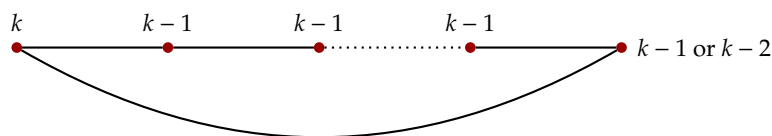
- (a) Prove that the last nonzero term in the chromatic polynomial of G is the term whose exponent is the number of components of G .
- (b) Use part (a) to prove that if $p(k) = k^n - ak^{n-1} + \dots \pm ck^r$ and $a > \binom{n-r+1}{2}$, then p is not a chromatic polynomial. (For example, this immediately implies that the polynomial in Exercise 5.3.3 is not a chromatic polynomial.)

Bonus question

7. Explain where the formula $\chi_{C_n}(k) = (k-1)^n + (-1)^n(k-1)$ comes from. (See **Problem 5.3.4** above.)

Solution.....

First we should understand why this computation is not trivial. The labels refer to how many colors are possible on that vertex. If we color them in sequence (like we would do with a path), then there are two possibilities for the last vertex, depending on whether its two neighbors are the same or different colors. The difficulty is counting how often each of these happens.



Look at some small concrete examples to get the pattern. The base case of the chromatic recurrence for cycles is $n = 3$.

$$\chi_{C_3}(k) = k(k-1)(k-2)$$

Then, for $n = 4$ we get

$$\begin{aligned}\chi_{C_4}(k) &= \chi_{P_4}(k) - \chi_{C_3}(k) \\ &= k(k-1)^3 - k(k-1)(k-2)\end{aligned}$$

now factor out a k , but not $(k - 1)$, to get an alternating power series:

$$\begin{aligned} &= k[(k - 1)^3 - (k - 1)((k - 1) - 1)] \\ &= k[(k - 1)^3 - (k - 1)^2 + (k - 1)] \end{aligned} \quad (1)$$

Now the tricky part: $k = (k - 1) + 1$. This will shift the power series and add this to the original, yielding cancellations:

$$\begin{aligned} &= ((k - 1) + 1)[(k - 1)^3 - (k - 1)^2 + (k - 1)] \\ &= (k - 1)^4 - (k - 1)^3 + (k - 1)^2 \\ &\quad + (k - 1)^3 - (k - 1)^2 + (k - 1)^1 \\ &= (k - 1)^4 + (k - 1) \end{aligned} \quad (2)$$

At this point a wrong guess would be that the pattern is $(k - 1)^n + (k - 1)$. But since we have an alternating sum we will do at least one more iteration ($n = 5$).

$$\chi_{C_5}(k) = \chi_{P_5}(k) - \chi_{C_4}(k) \quad (3)$$

Recursively substitute in (2),

$$\begin{aligned} &= k(k - 1)^4 - [(k - 1)^4 + (k - 1)] \\ &= (k - 1)^4(k - 1) - (k - 1) \\ &= (k - 1)^5 - (k - 1) \end{aligned}$$

But maybe we get more insight into “why” this formula is coming up by substituting (1) instead of (2) into (3),

$$\begin{aligned} &= k(k - 1)^4 - k[(k - 1)^3 - (k - 1)^2 + (k - 1)] \\ &= k[(k - 1)^4 - (k - 1)^3 + (k - 1)^2 - (k - 1)] \\ &= ((k - 1) + 1)[(k - 1)^4 - (k - 1)^3 + (k - 1)^2 - (k - 1)] \\ &= (k - 1)^5 - (k - 1)^4 + (k - 1)^3 - (k - 1)^2 \\ &= (k - 1)^4 - (k - 1)^3 + (k - 1)^2 - (k - 1) \\ &= (k - 1)^5 - (k - 1) \end{aligned}$$

Now we see better that the length of the alternating sum determines the \pm for the last term, and can make the proper conjecture *that you still need to prove*.

You will find 4 different proofs of this formula in this random paper: The Chromatic Polynomial for Cycle Graphs, [https://www.cs.tufts.edu/comp/150GT/documents/The Chromatic Polynomial for Cycle Graphs.pdf](https://www.cs.tufts.edu/comp/150GT/documents/The%20Chromatic%20Polynomial%20for%20Cycle%20Graphs.pdf). ♠

Project

Proposal due: Wed, 15 Mar 2023
Project due: Fri, 28 Apr 2023
Presentations: Mon, 1 May 2023, 6-8pm

The project is intended to get you to do/learn something independently, beyond what we have done in class. The project can be

INDIVIDUAL or TEAMS of 2.

First, you need to submit a

0. **PROPOSAL:** 1-sentence (or 1-paragraph) description of your project (email/txt)

The project will consist of

1. **REPORT:** (approx) 2-page paper on your topic (pdf)

Should include relevant definitions and examples, an explanation of the topic, the mathematical problem/question it addresses, the mathematical history of the problem/question, and connections with other mathematical topics (if any), and of course, a discussion of your specific topic.

2. **SLIDE PRESENTATION:** 5 minutes per person in team (pdf, ppt, or key)

Both your report and slides due Friday, 28 Apr 2023, but you can submit again on Monday if you make updates over the weekend before presentations.

Remember:
EXAM 2 is
Wed, Apr 26.

Presentations on Monday, 1 May 2023, 6:00-8:00pm. Please plan for that longer class time. *Please let me know if that timing is a problem for you.* There will be pizza.

Suggested topics

Pretty much any topic or algorithm of interest to you is ok, but it might be **better to choose a more specific topic** rather than a broad one.

1. First, check out *Quanta Magazine* <https://www.quantamagazine.org>. All of their articles are short and minimally technical, but always interesting. Do a search for "graph theory": [https://www.quantamagazine.org/search?q\[s\]=graph%20theory](https://www.quantamagazine.org/search?q[s]=graph%20theory) to see what's been in the headlines for the past few years.
2. Second, do an initial exploration of a topic using Wikipedia. Their math articles are often pretty good.
3. Do a search on arXiv (<https://arxiv.org>) to find research papers on your topic.
4. Don't hesitate to talk to me if you want a suggestion, or want to discuss any topic.
5. Below is a *dynamic* list of random ideas. I will keep adding to it. **Mostly, these are buzzwords to help you get your search for a topic started.**

-
- graph coloring
 - spectral graph theory
Spielman is the real deal. This talk discusses how algebraic graph theory touches on many topics in graph theory: <https://www.youtube.com/watch?v=CDMQR422LGM>
 - random graphs
 - something in computational geometry (Delaunay, art gallery, ?)
 - reconstruction conjecture
 - expander graphs
 - hypergraphs
 - network flows
 - graph isomorphism problem
 - <https://cacm.acm.org/magazines/2020/11/248220-the-graph-isomorphism-problem/abstract>
 - <https://arxiv.org/abs/1512.03547>
 - the graph minor theorem (Robertson-Seymour theorem)

Generally considered to be the most difficult result in all of graph theory. Lovász (another superstar) gives an intro and survey: <https://www.ams.org/journals/bull/2006-43-01/S0273-0979-05-01088-8/S0273-0979-05-01088-8.pdf>

Erik Demaine was only 20 when he became professor at MIT (the youngest every at MIT): https://erikdemaine.org/papers/Decomposition_FOCS2005/paper.pdf

- graph drawing

Start the video at 7:00 minutes to see an implementation of Tutte's theorem in graph drawing: <https://www.youtube.com/watch?v=CDMQR422LGM>

Force-Directed Drawing Algorithms: <https://cs.brown.edu/people/rtamassi/gdhandbook/chapters/force-directed.pdf>

Planar Straight-Line Drawings and Algorithms: <https://cs.brown.edu/people/rtamassi/gdhandbook/chapters/straightline.pdf>, e.g.,

- W. Schnyder. Embedding planar graphs on the grid. In Proc. 1st ACM-SIAM Sympos. Discrete Algorithms, pages 138–148, 1990.
- H. de Fraysseix, J. Pach, and R. Pollack. How to draw a planar graph on a grid. *Combinatorica*, 10(1):41–51, 1990.

- realizing an arbitrary group with autos of 3-regular graph
- tensor products of graphs and Shitov's counterexample to Hedetniemi's conjecture.

Project proposals submitted:

- graceful graphs
Claimed by MW. Another student **could** also do this topic, but we would just have to make sure your specific choices don't overlap too much.

Of course, take a look at the two papers I had already posted (especially Gallian):

<https://www.combinatorics.org/ojs/index.php/eljc/article/view/DS6graceful-labeling-algorithms-and-complexity.pdf>

Here are a couple dissertations that should include good overviews, as well as interesting stuff on more specific parts of the field:

<https://dsc.duq.edu/cgi/viewcontent.cgi?article=1625&context=etd>

<https://scholarworks.wmich.edu/cgi/viewcontent.cgi?article=4310&context=dissertations>

https://www.researchgate.net/publication/340306660_The_Graceful_Chromatic_Number_for_Some_Particular_Classes_of_Graphs

- Ringel's Conjecture
Claimed by AS
<https://arxiv.org/pdf/2001.02665.pdf>
<https://www.quantamagazine.org/mathematicians-prove-ringels-graph-theory-conjecture-20220511/>
- compiler register allocation (as a graph-coloring problem)
Claimed by JG
<https://www.sciencedirect.com/science/article/pii/S0096055181900485>
<https://dl.acm.org/doi/pdf/10.1145/74818.74843>
- Application: graph theory in video games
Claimed by TB.
- Decompositions
Claimed by VC.
<https://kostochk.web.illinois.edu/docs/2016/joc13-jw.pdf>
- Application: graph theory in neural networks
Claimed by GH/SZ.
- Edge colorings and mathematical origami
Claimed by JF.
<https://arxiv.org/abs/1910.05667>
- Art gallery theorems
Tentatively claimed by RR.
<https://www.science.smith.edu/~jorourke/books/ArtGalleryTheorems/art.html>
- Four color problem
Tentatively claimed by RR.

Homework 7

Due: Fri, 17 Mar 2023

1. **Problem 6.1.6.** (–) Prove that a plane graph is 2-connected if and only if for every face, the bounding walk is a cycle.

Solution.....

This one seemed pretty hard given that it was marked “easy” in the text. Here is the half we worked on in OH. The idea was to prove the contrapositive, but avoid all the special cases that we encountered in OH ...

(\Rightarrow). Prove the contrapositive. Assume there is a face F with a boundary walk $W = (v_0, e_1, v_1, \dots, e_k, v_k)$ that is not a cycle. Since W is not a cycle it must have at least one vertex that is revisited in the walk (other than being the first and last vertex of the walk). We want to show this is a cut-vertex.

Clearly, we can find a proper closed subwalk Y of W such that $Y = (v, \dots, v)$, and no vertex y is repeated along the subwalk Y (except for the two occurrences of v as the first and last vertex). Let X be the walk $W - Y$, with start and endpoint v .

If there are vertices $x \neq v$ on X , $y \neq v$ on Y , such that every xy -path in G goes through v , then v is a cut-vertex in G , and we are done. So assume this fails. Let $x = v_{i-1}$, $e = e_{i-1}$, $e' = e_{i+1}$, $y = v_{i+1}$, i.e., $W = (\dots, x, e, v, e', y, \dots, v, \dots)$. and let P be an xy -path that does not contain v . Clearly, P cannot be contained in W .

If e has F on both sides, then P would split F , contradiction. Similarly for e' .

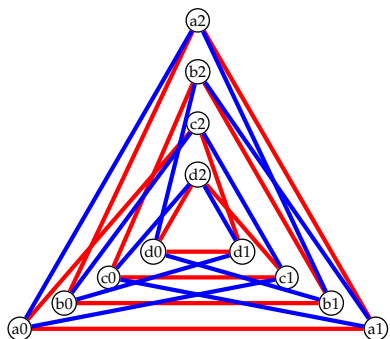
So e, e' can each have F on only one side. Now P gives a cycle Pvx whose interior is in the complement of F (since P not contained in W). Hence, W cannot cross P to get back to v .

By choice of Y , the part of W after y must revisit v before it gets back to x . This yv -walk will contain a yv -path P' , which will give a cycle $C = vP'$ whose interior intersects F . But this will split F since x is not on C , contradiction. \square

Definition. A graph is **outerplanar** if it has an embedding with every vertex on the boundary of the unbounded face.

2. **Problem 6.1.7.** (–) A **maximal outerplanar graph** is a simple outerplanar graph that is not a spanning subgraph of a larger simple outerplanar graph. Let G be a maximal outerplanar graph with at least three vertices. Prove that G is 2-connected.

3. **Problem 6.1.13.** Find a planar embedding of the graph below. *Note: You must explain how you got your planar embedding. The text gives an answer without explanation. . . My answer does not look like the text answer, but I give an explanation of how I got it.*

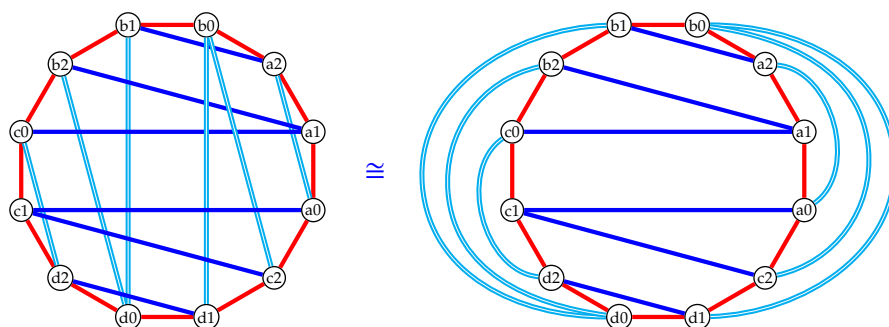


Solution.....

First, note that the answer in the text solution manual is incorrect—it doesn't even have the correct number of edges. However, it is really just a graphical "typo" and is easy to fix . . .

The graph is even, in fact 4-regular, so it is Eulerian and decomposes into cycles. In fact, it decomposes into two (Hamiltonian) cycles, and is hence planar, i.e., draw the vertices on a circle and put one cycle inside and one cycle outside the circle.

If we put these vertices on a circle (in the order of the red cycle) we get:



Note that the **blue** cycle can be partitioned into two halves as indicated by the **blue/cyan** (i.e., "horizontal"/"vertical") coloring above. It is easy to see that either half (blue or cyan) can be drawn outside the red cycle without edge crossings, thus yielding a planar graph. \square

- **Problem 6.1.21.** (!) Prove that a set of edges in a connected plane graph G forms a spanning tree of G if and only if the duals of the remaining edges form a spanning tree of G^* . *Don't turn in, but take a look at this one. We didn't talk about spanning trees in class, but this shows a nice connection between duals and spanning trees.*

4. **Problem 6.1.25.** (!) Prove that every n -vertex plane graph isomorphic to its dual has $2n - 2$ edges. For all $n \geq 4$, construct a simple n -vertex plane graph isomorphic to its dual.

Bonus problem

5. Let G be connected planar. Find a condition on G that ensures G^* is simple.

Solution


If G has a cut-edge, then G^* will not be simple—it will have a loop.

If G has a edges u, v such that removal of only one leaves G connected, but removal of both disconnects G .

So our condition is for G to be 3-connected. □

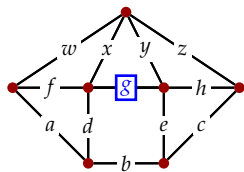
Quiz 4

Wed, 15 Mar 2023

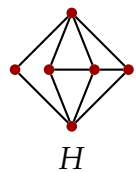
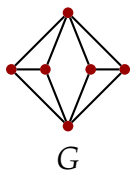
1. Let $G =$ . Compute: $\chi_G(k) =$ $k(k-1)^2(k-2)^2$. *Hint: no need for chromatic recurrence.*

2. The largest number of colors the greedy algorithm would ever use on a complete binary tree of height h is 4. *Note: "complete" and "height h " are both irrelevant.*

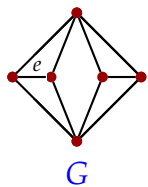
3. Removing (horizontal) edge g in graph G below gives a 4-critical subgraph.



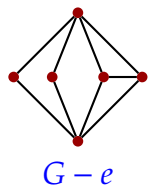
4. Do *not* compute χ_G or χ_H , but give a short proof that they are equal.



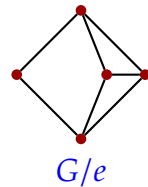
Solution.....



\Rightarrow



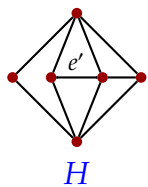
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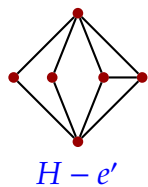
$$\chi_G(k) = \chi_{G-e}(k) - \chi_{G/e}(k)$$

$$= \chi_{H-e}(k) - \chi_{H/e}(k)$$

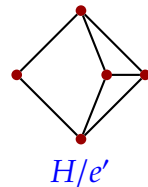
$$= \chi_H(k)$$



\Rightarrow



$-$



Homework 8

Due: Fri, 31 Mar 2023

Not graded. Do not turn in.
(But there still ~~might~~ will be a quiz on Wed, Apr 5)

1. **Problem 6.2.2.** (–) Give three proofs that the Petersen graph is nonplanar.

(a) Using Kuratowski's Theorem.

Solution

Find a subdivision of $K_{3,3}$.

(b) Using Euler's Formula and the fact that the Petersen graph has girth 5.

Solution

Assume a planar embedding. Then each face has ≥ 5 edges on its boundary, so $2e \geq 5f$. Substitute $f \leq \frac{2}{5}e$ into $n - e + f = 2$ to get a contradiction.

(c) Using the planarity-testing algorithm of Demoucron-Malgrange-Pertuiset.

Skip this part, but you might be interested in taking a look in the text . . .

2. **Problem 6.2.6.** (!) Fáry's Theorem. Let R be a region in the plane bounded by a simple polygon with at most five sides (simple polygon means the edges are line segments that do not cross). Prove there is a point x inside R that "sees" all of R , meaning that the segment from x to any point of R does not cross the boundary of R . Use this to prove inductively that every simple planar graph has a straight-line embedding.

Solution

i) For the polygons with ≤ 5 sides, just do a case analysis. For 1 side there is one case, for 4 sides there are two cases, and for 5 sides there are four cases.

ii) Then recursively build the desired embedding. Idea (1): can replace the given embedding a G and triangulate it. Idea (2): as seen before, there is a vertex $v \in G$ with $\deg(v) \leq 5$. Recurse on this one by noting that the face of $G - x$ containing x is a cycle, then replacing x at the point found in part (i).

3. **Problem 6.3.1.** (–) State a polynomial-time algorithm that takes an arbitrary planar graph as input and produces a proper 5-coloring of the graph.

Solution

Basic idea: show that processing via Kempe-chains is polynomial.

4. **Problem 6.3.3.** (–) Use the Four Color Theorem to prove that every outerplanar graph is 3-colorable.

Solution.....

Let G be simple outerplanar, and v be a new vertex. Then $G \vee v$ is simple planar, hence 4-colorable. But only 3 colors can be used on G , and the 4th color is only used on v .

5. **Problem 6.3.5.** Prove that every planar graph decomposes into two bipartite graphs. (Hedetniemi [1969], Mabry [1995])
Hint: Use the 4-color Theorem.

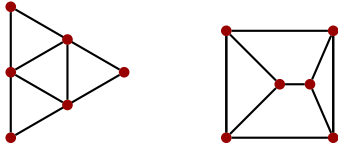
Solution.....

Any pair of colors from a 4-coloring will give a bipartite graph.

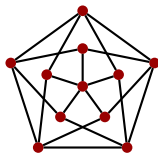
Homework 9

Due: Fri, 7 Apr 2023

1. **Problem 7.1.1.** (–) For each graph G below, compute $\chi'(G)$ and draw $L(G)$.



2. **Problem 7.1.3.** (–) Determine the edge-chromatic number of $C_n \square K_2$.
3. **Problem 7.1.5.** (–) Prove that the Petersen graph is the complement of $L(K_5)$.
4. **Problem 7.1.24.** (–) Let G and H be nontrivial simple graphs. Use Vizing's Theorem to prove that $\chi'(H) = \Delta(H)$ implies $\chi'(G \square H) = \Delta(G \square H)$.
5. **Problem 7.2.2.** (–) Is the Grötzsch graph (Example 5.2.2 in the text, shown below) Hamiltonian?



6. **Problem 7.2.7.** A mouse eats its way through a $3 \times 3 \times 3$ cube of cheese by eating all the $1 \times 1 \times 1$ subcubes. If it starts at a corner subcube and always moves on to an adjacent subcube (sharing a face of area 1), can it do this and eat the center subcube last? Give a method or prove impossible. (Ignore gravity.)

Quiz 5

Wed, 5 Apr 2023

1. Let G be a simple planar bipartite graph on 6 vertices. What is the maximum number of faces that a planar embedding of G can have? You must give a (simple) computation/explanation that verifies your answer (not brute force) below.

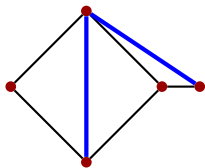
$$\max f = \boxed{4}.$$

Solution.....

By Euler's formula $n - e + f = 2$, we get $f = e - 4$. To maximize this (and still have a connected planar graph) we set $e = 8$ to get $f = 4$.

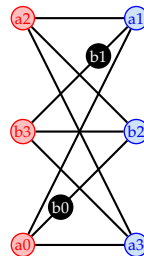
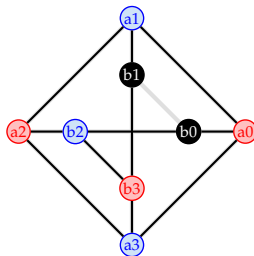
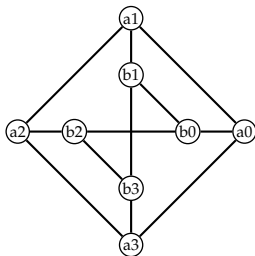
2. A **maximal outerplanar graph** has an embedding with every vertex on the boundary of the unbounded face, and is not a spanning subgraph of a larger simple outerplanar graph.

How many edges must be added to the graph below for it to be maximal outerplanar? If your answer is > 0 , add them to the picture below.



add $\boxed{2}$ edges

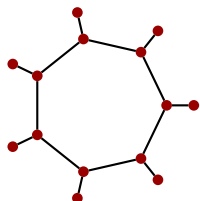
3. Give a planar embedding of the graph below if possible. Otherwise, find a Kuratowski subgraph on the template provided. (Maybe use different shapes for vertices of different colors.)



A subdivision of $K_{3,3}$

4. Let G_n be built from C_n and n new vertices with each node of C_n adjacent to exactly one

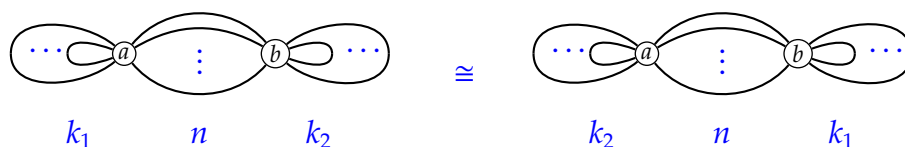
of the new nodes, so $|V(G_n)| = |E(G_n)| = 2n$. One planar embedding of G_7 is pictured below. How many duals (distinct up to isomorphism) does (the abstract graph) G_n have?



number of duals of $G_n = \left\lceil \frac{n+1}{2} \right\rceil$ (arbitrary n)

Solution.....

The iso type of the dual depends only on how many spikes are on different sides of the cycle. If there are k_1 on one side, and $k_2 = n - k_1$ on the other, then we have k_1 self-loops on a , n multiple edges between a, b , and k_2 loops on b .



Furthermore, k_1 spikes on the inside and k_2 on the outside gives the same dual as (i.e., is isomorphic to) k_1 on the outside and k_2 on the inside (where of course, $k_1 + k_2 = n$).

For example, $n = 3 \Rightarrow$ the (in, out)-possibilities are $(0, 3), (1, 2), (2, 1), (3, 0)$, but only 2 distinct duals (up to isomorphism). For $n = 4$ the (in, out)-possibilities are $(0, 4), (1, 3), (2, 2), (3, 1), (4, 0)$, and 3 distinct duals. In general,

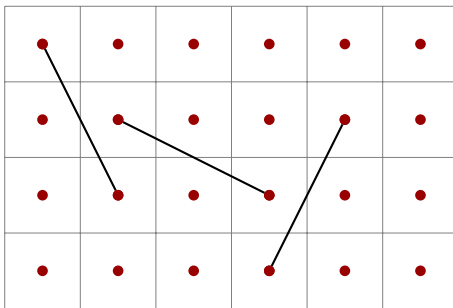
$$\text{the number of distinct duals} = \begin{cases} \frac{n+1}{2} & \text{if } n \text{ is odd} \\ \frac{n}{2} + 1 & \text{if } n \text{ is even} \end{cases}$$

$$= \left\lceil \frac{n+1}{2} \right\rceil.$$

Homework 10

Due: Fri, 14 Apr 2023

1. **Problem 7.2.8.** (!) On a chessboard, a **knight** can move from one square to another that differs by 1 in one coordinate and by 2 in the other coordinate, as shown below. Prove that no $4 \times n$ chessboard has a **knight's tour**: a traversal by knight's moves that visits each square once and returns to the start. (Hint: Find an appropriate set of vertices in the corresponding graph to violate the necessary condition.)



Solution.....

Some simple ideas that might lead you to the solution:

- (a) Note that the graph is bipartite. Then you will want to remove nodes of only one color (if possible), in order to minimize the number you need to remove.
 - (b) Remove nodes in the middle since they have higher degree than those on the edges.
2. **Problem 7.2.30.** Obtain Lemma 7.2.9 (sufficiency of Ore's condition) from Theorem 7.2.13 (sufficiency of Chvátal's condition). (Bondy [1978])

Solution.....

Dirac's condition: for all u , $\deg(u) \geq n/2$ (1)

Ore's condition: u, v not adjacent $\Rightarrow \deg(u) + \deg(v) \geq n$ (2)

Chvátal's condition: $i < n/2 \Rightarrow d_i > i$ or $d_{n-i} \geq n - i$ (3)

Note: we trivially have, (1) \Rightarrow (2), hence, Ore's Theorem \Rightarrow Dirac's Theorem, i.e., "(2) \Rightarrow Hamiltonian" \Rightarrow "(1) \Rightarrow Hamiltonian".

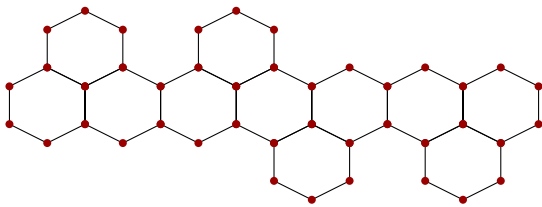
We want to prove Ore's Theorem " $(2) \Rightarrow \text{Hamiltonian}$ ", by assuming Chvatal's Theorem " $(3) \Rightarrow \text{Hamiltonian}$ ". Clearly it is enough to show $(2) \Rightarrow (3)$.

3. **Problem 7.2.31.** (!) Prove or disprove: If G is a simple graph with at least three vertices, and G has at least $\alpha(G)$ vertices of degree $n(G) - 1$, then G is Hamiltonian.
4. **Problem 3.1.8.** (!) Prove or disprove: Every tree has at most one perfect matching.
5. **Problem 3.1.9.** (!) Prove that every maximal matching in a graph G has at least $\alpha'(G)/2$ edges.

Homework 11

Due: Fri, 21 Apr 2023

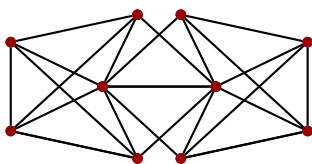
- Problem 3.1.11.** Let C and C' be cycles in a graph G . Prove that $C \Delta C'$ decomposes into cycles.
- Problem 3.1.28.** (!) Exhibit a perfect matching in the graph below or give a short proof that it has none. (Lovász-Plummer (1986, p7])



- Problem 4.1.1.** (–) Give a proof or a counterexample for each statement below.
 - Every graph with connectivity 4 is 2-connected.
 - Every 3-connected graph has connectivity 3.
 - Every k -connected graph is k -edge-connected.
 - Every k -edge-connected graph is k -connected.

Don't turn in, but take a look at:

- Problem 4.1.8.** Determine $\kappa(G)$, $\kappa'(G)$, and $\delta(G)$ for each graph G drawn below.



Quiz 6

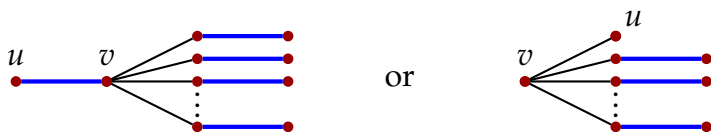
Fri, 21 Apr 2023

1. Let P be the Petersen graph. The edge-chromatic number $\chi'(P \square C_8) =$ 5.

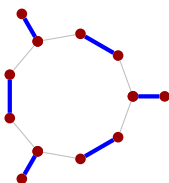
Solution.....

$\chi'(P) = 4$, but by HW 9.4, $\chi'(C_8) = \Delta(C_8) = 2 \Rightarrow \chi'(P \square C_8) = \Delta(P) + \Delta(C_8) = 3 + 2 = 5$.

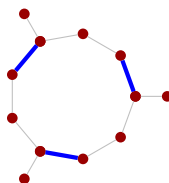
2. Show that for every $k \geq 3$, there is a tree T with a leaf node u that is adjacent to a vertex v of degree k , and that $T - \{u, v\}$ does have a (non-empty) perfect matching. Modify the given pic.



3. For maximal matching M , $|M| \geq \alpha'(G)/2$. Give an example to indicate there are arbitrarily large graphs witnessing this bound to be tight.



$$\alpha' =$$
 6



$$|M| =$$
 3

4. A new chess piece, a **Knight-King** can move one square in any direction, or move as a knight. There is obviously a Knight-King tour on a 4×4 board. Explain which of the following (if any) guarantee a Hamiltonian cycle. *Hint: Indicate the degrees of each node on the board.*

8	12		
5	8		

(a) Dirac: **NO**: $\delta = 5 < n/2 = 8$

(b) Ore: **NO**: corners are not adjacent, but $5 + 5 < 16$

(c) Chvatal: **YES**: for all $i < n/2 = 8$, $d_i > i$
 $(5, 5, 5, 5, 8, 8, 8, \dots)$

Exam 2 Hints

The exam is on **Wed, 26 Apr 2023** in class. No notes or books, but I will supply any definitions you ask for.

When building the exam, I will be looking at

1. quizzes
2. homework
3. problems in the text marked with (-), i.e., the easy ones

3:30pm Tuesday, 25 April 2023

The exam is not finished, but I have problems on these topics:

Kempe chains

chromatic recurrence formula

Euler

Kuratowski's theorem

compute edge-chromatic number

dual graph

matchings/Hall's condition

connectivity of a graph

Hamiltonian cycle

Exam 2

Wed, 26 Apr 2023

Name on every leaf. Please print your name 4 times (once on each leaf).

Use the answer boxes.

Show your work.


All problems of equal value. All parts of a problem are of equal value.

Work exactly **8** of the problems.

On the 4 problems that you don't work place a **large visible** star \star .

You can only assign a \star to an entire problem, not just a part.

You **must** assign 4 \star 's.

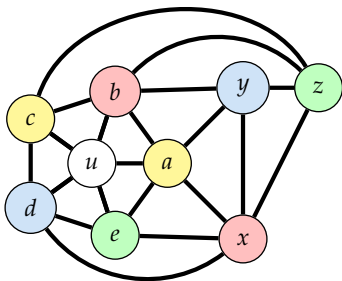
1. A difficult problem...

2. An easy problem ...
my correct solution
⋮

Good luck!

Typos fixed in class: #3, #7a, #9b, #11 (corrected in this version)

Kempe originally believed he could prove the 4-color theorem, but Heawood showed his technique was only strong enough for the 5-color theorem.

1. The graph below is planar, so it *does* have a 4-coloring using only the colors $\{R, G, B, Y\}$. Let H_{XY} = the subgraph induced by the vertices colored X or Y , for $X, Y \in \{R, G, B, Y\}$. Show which (if any) of the H_{XY} can be used to color vertex u , thus giving a 4-coloring of the graph.



- (a) RG: NO: nodes b, e are connected in H_{RG} , so switching leaves 4 colors in $N_G(u)$
- (b) RB: NO: nodes b, d are connected in H_{RB}
- (c) RY: NO: nodes a, b are connected and b, c are connected in H_{RY}
- (d) BG: NO: nodes d, e are connected in H_{BG}
- (e) BY: NO: switching in either or both components of H_{BY} leaves 4 colors on $N_G(u)$
- (f) GY: NO: switching in either or both components of H_{GY} leaves 4 colors on $N_G(u)$

2. Let $G = K_n \square C_{2n}$. Then $\chi'(G) = \boxed{n + 1}$.

Solution.....

$$\chi'(C_{2n}) = \Delta(C_{2n}) = 2 \quad \text{and} \quad \Delta(K_n) = n - 1 \quad \Rightarrow \quad \chi'(G) = n - 1 + 2 = n + 1. \quad \text{☺}$$

3. Fix simple connected graph G with chromatic polynomial $\chi_G(k)$. Build a new graph G' by doing the following: for each edge $e = uv \in E(G)$, add a new vertex $w_e \notin V(G)$, and two new edges uw_e, vw_e . As usual, let n be the number of vertices, and m the number of edges. So $n' = n + m$ and $m' = 3m$. Compute $\chi_{G'}(k)$ in terms of n, m , and $\chi_G(k)$.

$$\chi_{G'}(k) = \boxed{(k-2)^m \cdot \chi_G(k)}$$

Solution.....

Adding a single “ear” to get (say) G_1 gives

$$\begin{aligned}\chi_{G_1}(k) &= \chi_{G+uw_e}(k) - \chi_G(k) \\ &= (k-1) \cdot \chi_G(k) - \chi_G(k) \\ &= (k-2) \cdot \chi_G(k)\end{aligned}$$

Hence, adding all m -many of the ears gives

$$\chi_{G'}(k) = (k-2)^m \cdot \chi_G(k)$$

☺

4. Let Q_n = the hypercube of dimension n . Then $\chi'(Q_n) = \boxed{n}$.

Solution.....

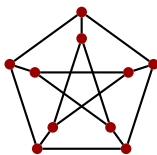
$$Q_n = P_2 \square Q_{n-1}, \chi'(P_2) = 1 = \Delta(P_2), \text{ so } \chi'(Q_n) = 1 + \chi'(Q_{n-1}) = n.$$

☺

5. The maximum number of faces for a planar subgraph of the Petersen graph P is

5

Solution.....



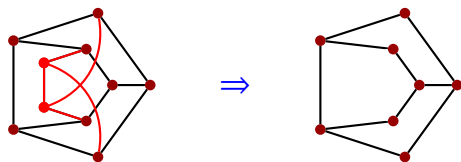
For the subgraph we will have $n - e + f = 2$, so $f = 2 + e - n$. We want n small and e large.

Note: It would really help to know that P is both vertex-transitive and edge-transitive. Since all vertices “look the same” and all edges “look the same”, this reduces the number of cases.

Removing vertices:

Note that if we remove one vertex from P , we get a nonplanar subgraph, so we must remove at least two vertices from P .

If we remove two adjacent vertices we get a planar subgraph, so $f = 2 + 10 - 8 = 4$.



If we remove two nonadjacent vertices, we get $f = 2 + 9 - 8 = 3$.

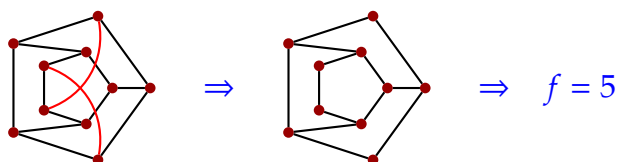
If we remove no vertices, but only edges, then

Removing edges:

If we remove 1 edge, we get a nonplanar subgraph.

If we remove 2 edges, it is *possible* to get a planar subgraph. (These must be nonincident.)

This gives our answer: $f = 2 + 13 - 10 = 5$.



6. Use Kuratowski's theorem to prove:

G is outerplanar if and only if G contains no subdivision of K_4 or $K_{2,3}$.

Hint: Add a new vertex $x \notin V(G)$ in the unbounded face of G . Let $H = G \vee x$, i.e., add an edge xv , for every $v \in V(G)$.

Solution

Note: H planar iff G outerplanar.

G outerplanar $\Rightarrow H$ planar $\Rightarrow H$ has no $K_5, K_{3,3} \Rightarrow G$ has no $K_4, K_{2,3}$

G not outerplanar $\Rightarrow H$ not planar $\Rightarrow H$ has either $K_5, K_{3,3} \Rightarrow G$ has either $K_4, K_{2,3}$ ☺

7. The term "dual" is usually reserved for the case where the dual is its own inverse, i.e., the dual of the dual is the original object.

(a) Give an example of a planar G with 3 vertices for which $G^{**} \neq G$.

Solution

 has the dual  ☺

(b) Hence, we get $G^{**} \cong G$ if G is connected. (No need to give a proof.)

8. Let G be a k -regular X, Y -bigraph. Show that Hall's condition is satisfied (hence, G has a perfect matching). *Hint: Compare the number of edges out of a set $S \subseteq X$ to the number of edges out of $N(S)$.*

Solution

Fix arbitrary $S \subset X$. Recall that $G[S]$ = the subgraph induced by S . Then

$$\underbrace{|E(G[S])|}_{\text{edges out of } S} = k \cdot |S| \quad \text{and} \quad \underbrace{|E(G[N(S)])|}_{\text{edges out of } N(S)} = k \cdot |N(S)|.$$

Clearly, the first is \leq the second, so $k|S| \leq k|N(S)|$, hence $|S| \leq |N(S)|$ (which is Hall's condition). ☺

9. $\kappa(G)$ = the minimum size of a vertex set S such that $G - S$ is disconnected or has only one vertex.

(a) Show $\kappa(G) \leq \delta(G)$.

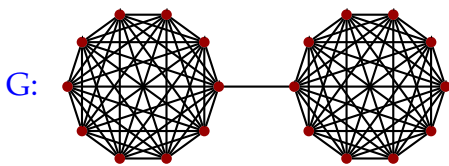
Solution

Let $\deg(v) = \delta(G)$. Then $G - N(v)$ will have at least one more component than G , comprised of v , so $\kappa \leq \delta$. (Note that $(G - N(v)) - v \neq \emptyset$ because each $w \in N(v)$ has δ -many neighbors.) ☺

- (b) Give a family of connected graphs to show that the difference $\delta - \kappa$ can be arbitrarily large.

Solution

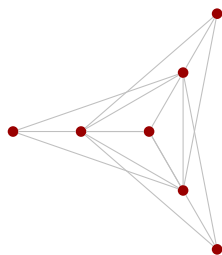
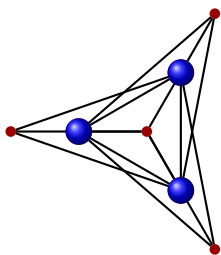
Consider the disjoint union $K_n + K_n$, for $n \geq 3$. Then add one new edge between a single vertex in each copy of K_n to get a connected graph G .



Then $\kappa(G) = 1 < \delta(G) = n - 1$

☺

10. Find a Hamiltonian cycle or show that there is not one.



Removing the 3 vertices of degree 6 leaves 4 components. Since $4 > 3$, the graph is not Hamiltonian.

11. A 2-King move consists of two legal King moves done in succession. What is the largest n so that Chvatal's condition guarantees a Hamiltonian cycle of 2-King moves on an $n \times n$ board?

	15		
8	11		

		24	
	15	19	
8	11	14	

		24		
	15	19		
8	11	14		

$$n = \boxed{5}$$

$n = 4 \Rightarrow 8, 8, 8, 8, 11, 11, 11, 11, 11, 11, 11, 11, 15, 15, 15, 15$

$n = 5 \Rightarrow 8, 8, 8, 8, 11, 11, 11, 11, 11, 11, 11, 14, 14, 14, 14, 15, 15, 15, 15, 19, 19, 19, 19, 24$

$n = 6 \Rightarrow d_1 - d_4 = 8, d_5 - d_{12} = 11, d_{13} - d_{16} = 15, d_{17} - d_{24} = 14, d_{25} - d_{32} = 19, d_{33} - d_{36} = 24$
 $\Rightarrow d_{11} = 11 \leq 11$ and $d_{25} = 19 < 25$, so Chvatal's condition not satisfied.

12. The following was posted to stackoverflow. Please help this programmer out by answering their question.

Knight's Tour on a 5 x 5 Board Start from any Square? Asked 7 years, 9 months ago, Modified 9 months ago, Viewed 5k times

I'd just like to check my logic here...

I wrote code to solve the Knight's Tour and it works well for 8x8 boards starting the Knight at any square.

But... on a 5x5 board I show no solution possible when starting at square (0, 1).

What I tried for 5x5 starting the Knight at Row 0, Col 1:

Warnsdorff's path Added Roth (tie breakers based on Euclidean distance from center).

Since those did not produce a solution I did code that is just basic recursion with backtracking to test every possible path -- also no solution found when starting a 5x5 on 1, 0.

I looked everywhere for a list of exhaustive solutions to the 5x5 board but found none.

Is it that there just is no solution for 5x5 when starting at square 0, 1?

Thank you!

Is there a Hamiltonian cycle? **NO**

Solution.....

Bipartite graph, but a Hamiltonian cycle would have an odd number (25) of vertices.☺

Final Project Presentations

Vote your favorites. The most votes win fabulous non-cash prizes!

Presenter	Title	Distribute 10 points among 9 candidates
Sabrina	Hypergraphs: Modeling Complex Interactions	
Avi	Ringel's conjecture	
Jasper	Register allocation by graph coloring	
Grace and Sachin	Neural Networks on Graphs and Hypergraphs	
Matthew	Graceful Labelings for Trees	
Raymond	Four colors suffice	
Johnna	Where Graph Theory and Origami Overlap	
Tyler	Graph Theory and Games	
Vera	Decompositions of Cartesian Products of Complete Graphs into Sunlet Graphs of Order 8	

Total = 10 points