GRACEFUL LABELING ALGORITHMS AND COMPLEXITY - A SURVEY

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Abstract. Graceful graphs were first studied by Rosa in 1966. The Kotzig-Ringel graceful tree conjecture states that every tree has a graceful labeling. Aldred and McKay and others [6, 15, 20] have used computer programs to show that trees of order up to 35 are graceful. Bagga et al. investigated algorithms for generating all graceful labelings of certain known classes of graceful graphs, including paths, cycles, and certain other classes of unicyclic graphs. The data generated by such algorithms has led to the discovery of new properties of such graceful labelings. In this paper we present a survey of graceful graph labeling algorithms and related complexity issues.

Key words: Graph labelings, graceful labeling, algorithms, complexity.

Abstrak. Graf-graf graceful pertama kali dipelajari oleh Rosa pada tahun 1966. Konjektur pohon graceful Kotzig-Ringel menyatakan bahwa setiap pohon memiliki suatu pelabelan graceful. Aldred dan McKay (1998), Fang (2011), dan Horton (2003) telah menggunakan program komputer untuk menunjukkan bahwa pohon dengan orde hingga 35 adalah graceful. Bagga dkk. (2007) telah menyelidiki algoritma untuk menghasilkan semua pelabelan graceful dari graf-graf graceful untuk kelas-kelas graf tertentu seperti lintasan, lingkaran, dan kelas tertentu lainnya dari graf unicyclic. Data yang dihasilkan oleh algoritma tersebut menuntun pada penemuan sifat baru dari pelabelan graceful. Pada artikel ini kami menyajikan sebuah survei dari algoritma pelabelan graf graceful dan isu-isu kompleksitas terkait.

Kata kunci: Pelabelan graf, pelabelan graceful, algoritma, kompleksitas.

1. Introduction

For a graph G, we are interested in labeling the vertices and/or edges of G with nonnegative integers. More formally, a *labeling* of a graph $G = (V, E)$ is a one-one mapping $f : V \to X$, where X is a (finite) set of nonnegative integers. Such a vertex labeling f induces an edge labeling, where an edge $e = uv$ gets the label $|f(u) - f(v)|$. Suppose that G has q edges. We are interested in labelings where the set of possible labels X is $\{0, 1, 2, ..., q\}$ or $\{0, 1, 2, ..., 2q\}$. Rosa [23] called such a labeling a *valuation*. He considered four special types of valuations.

Let f be a valuation of a graph G with q edges.

(1) f is called an *α-valuation* if $X \subseteq \{0, 1, 2, ..., q\}$, the induced edge labels are $1, 2, ..., q$ and there exists an $x \in \{0, 1, 2, ..., q\}$ such that for any edge uv of G, either $f(u) \leq x < f(v)$ or $f(v) \leq x < f(u)$.

(2) f is called a *β-valuation* if $X \subseteq \{0, 1, 2, ..., q\}$ and the induced edge labels are $1, 2, ..., q$.

(3) f is called a *σ-valuation* if $X \subseteq \{0, 1, 2, ..., 2q\}$ and the induced edge labels are $1, 2, ..., q$.

(4) f is called a *ρ-valuation* if $X \subseteq \{0, 1, 2, ..., 2q\}$ and the induced edge labels are $x_1, x_2, ..., x_q$ where, $x_i = i$ or $x_i = 2q + 1 - i$ for each $1 \leq i \leq q$.

Observe that the above definitions are from the strongest to weakest in the sense that every α-valuation is also a β-valuation, every β-valuation is also a σ-valuation, and every σ-valuation is also a ρ-valuation. Golomb [18] introduced the term *graceful labeling* for a β-valuation. A *graceful* graph is one that has a graceful labeling. Rosa's paper [23] generated much research activity in this area and led to a large number of publications. A substantial part of this research effort has been devoted to finding new families of graceful graphs. Several variations of the concept have also been studied. We refer the reader to Gallian's ongoing survey [16] for an excellent account of research in this area. One of the foremost open problems is the Ringel-Kotzig conjecture that every tree is graceful. As noted by Gallian [16], Aldred and McKay proved, with the aid of a computer program, that all trees with at most 27 vertices are graceful. This has been extended to trees of order up to 35 [15, 20]. It is also known that paths, caterpillars and several other subclasses of trees are graceful.

In this paper we present a survey of graceful graph labeling algorithms and related complexity issues. In Section 2, we discuss computational efforts aimed at verifying the graceful tree conjecture. In Section 3, we look at complexity issues in graceful graphs. In Section 4, we survey recent results on generating all graceful labelings of certain families of unicyclic graphs. In Section 4, we discuss other computational and mathematical methods for determining gracefulness. Finally, we conclude in Section 5 with a summary and some open problems.

2. Graceful Tree Conjecture

Rosa considered the above valuations in order to investigate decompositions of graphs into trees. For the sake of completeness, we include a brief discussion here. A *decomposition* of a graph $G = (V, E)$ can be considered to be a partition of the edge set E into subsets $E_1, E_2, ..., E_r$. If H_i is the subgraph of G induced by E_i, then we also say the G decomposes into subgraphs $\{H_i\}$. If the subgraphs H_i are all isomorphic to a single graph (say) H then we say that G is $H - decomposable$ and we write $H|G$. In 1963, Ringel [22] made the following conjecture.

Conjecture 2.1. *If T is a tree with m edges, then K_{2m+1} decomposes into $2m+1$ copies of T.*

Rosa [23] proved the following theorem.

Theorem 2.2. *If a tree T with m edges has a graceful labeling, then K_{2m+1} decomposes into $2m+1$ copies of T.*

In fact, Rosa proved more. A decomposition of K_n by r copies of a subgraph H is called a *cyclic decomposition* if, when K_n is drawn appropriately with its vertices on a regular polygon, the r copies of H can be obtained by rotations of an appropriate copy of H. Rosa [23] proved the following results.

Theorem 2.3. *Let G be a graph with m edges. Then G has a ρ- valuation if and only if K_{2m+1} has a cyclic decomposition into $2m+1$ copies of G.*

Theorem 2.4. *Let G be a graph with m edges, and $k \geq 1$ be an integer. If G has an α- valuation then K_{2km+1} has a cyclic decomposition into $k(2km+1)$ copies of G.*

Conjecture 2.1 was followed by the following stronger conjecture, which is known as the Kotzig-Ringel graceful tree conjecture.

Conjecture 2.5. *Every tree has a graceful labeling.*

A substantial amount of research effort has been spent towards finding a proof of this conjecture. A number of special classes of trees have been shown to be graceful. These include paths, caterpillars, lobsters with perfect matchings [21, 23] and several others. Rosa [23] showed that all trees with less than 16 edges are graceful, and all trees of up to 4 leaves are graceful.

Theorem 2.6. [6, 15, 20] *Every tree of order up to 35 is graceful.*

Aldred and McKay used a computer search to prove the above result. They described this search algorithm as a combination of hill climbing and tabu search. In the same paper they also showed that all trees of at most 27 vertices are harmonious ([19]). Horton [20] used a random backtracking algorithm to show that trees of order up to 29 are graceful. To enumerate trees with a given number of vertices, the algorithm proposed by Wright et al [26] is used. Fang [15] describes a computational approach and shows that every tree of order up to 35 is graceful.

Verification of gracefulness for trees with 33, 34 or 35 vertices is accomplished with the help of a volunteer computing community, where the computational task is divided into small fragments that are carried out on heterogeneous machines. It is estimated that to verify the gracefulness on a single machine, it may take up to 7.7 years on a Core 2 Duo T7200 computer.

The following result of Erdös [18] is well-known. See [19] for a proof.

Theorem 2.7. *Almost all graphs are not graceful.*

However, the research on the Kotzig-Ringel graceful tree conjecture and on finding other families of graceful graphs continues.

3. Generating and Enumerating Graceful Labelings

We now discuss bounds on the numbers of graceful labelings for some classes of graphs. For paths, let $G(n)$ denote the number of graceful labelings of P_n. Aldred et al [7] proved the following result.

Theorem 3.1. $G(n) = \Omega((5/3)^n)$.

Adamaszek [5] studied graceful labelings in the context of permutations. He defined a *graceful n-permutation* as a permutation $[\sigma(0), ..., \sigma(n-1)]$ of the set $\{0, 1, ..., n-1\}$ such that $\{|\sigma(1) - \sigma(0)|, |\sigma(2) - \sigma(1)|, ..., |\sigma(n-1) - \sigma(n-2)|\} = \{1, ..., n-1\}$. For example, $[0, 6, 1, 5, 2, 4, 3]$ is a graceful n-permutation. Thus a graceful n-permutation of the set $\{0, 1, ..., n-1\}$ can be identified with a graceful labeling of P_n. Let $G(n)$ denote the number of graceful n-permutations. According to [5], the sequence $G(n)$ has number $A006967$ in Sloane's On-line Encyclopedia of Integer Sequences [24] where the first 20 terms are listed. Adamaszek generated and counted graceful n-permutations by a recursive search procedure, and improved Theorem 3.1 as follows.

Theorem 3.2. $G(n) = \Omega((2.7)^n)$.

Adamaszek [5] also states that the quotients $G(n+1)/G(n)$ tend to gather between 3 and 4.5, suggesting that the lower bound $(2.7)^n$ could be improved further.

Eshghi and Azimi [13, 14] investigated mathematical programming techniques to solve a model of the graceful labeling problem. They used a branch-and-bound algorithm to solve the corresponding integer programming problem. This technique was applied to find graceful labelings of randomly generated samples of several classes of graphs. Some of their computational results for trees are summarized in the table below. The last column shows the average time to generate a graceful labeling over a sample of thirty randomly generated trees. All computations were run on a Pentium IV 2500 MHz computer with 256 MB RAM. See [13, 14] for details.

Number of vertices	Average time (Seconds)
20	149.12
25	2898.14
28	3827.11
30	4427.25
35	7625.69
40	11321.54

Table 3.1: Computation time for trees (30 samples of each order)

We next discuss graceful labelings of certain classes of unicyclic graphs. We first look at cycles. Rosa [23] found necessary and sufficient conditions for cycles to be graceful.

Theorem 3.3. *The cycle C_n is graceful if and only if $n \equiv 0$ or 3 (mod 4).*

Since a unicyclic graph G on n vertices has n edges, it follows that if G has a graceful labeling, then exactly one of the integers from the set $\{0, 1, 2, \cdots, n\}$ is missing from the set of vertex labels. We denote this missing label by m. Bagga et al [8, 9, 10] designed algorithms for generating all graceful labelings of certain classes of unicyclic graphs. We include here a brief description of the algorithm that generates all graceful labelings of C_n, with $n \equiv 0$ or 3 (mod 4). We observe that vertex labels 0 and n must appear on adjacent vertices in any graceful labeling of C_n since this is the only way of obtaining edge label n. Our algorithm starts by labeling two adjacent vertices of C_n as 0 and n. It then exhaustively checks all possible ways of generating edge labels $n - 1$, $n - 2$,... in that order. We call the edge label being generated as the "level". This leads to a tree of computations. Along each branch of this tree, a possible graceful labeling is explored. A branch which reaches level 1 yields a graceful labeling. Table 3.2 shows how such branches start out from level n.

Level n		$0\ n$		
Level $n-1$	$n-1\ 0\ n$		$0\ n\ 1$	
Level $n-2$	$1\ n-1\ 0\ n$	$n-1\ 0\ n\ 2$	$n-2\ 0\ n\ 1$	$0\ n\ 1\ n-1$
Level $n-3$		...		
...		...		
Level $n-k$		...		

Table 3.2: Tree of computations

Traversal along each branch can be thought of as building a set of disjoint paths on the cycle, with the edges on the paths being those which have already been labeled. At level k, the edges on the paths have labels n, $n - 1$, ..., k. Depending on the vertex labels on these paths, there are three possible cases in which a new edge labeled k can be obtained. (i) Two end vertices on two existing paths are adjacent on the cycle and have absolute difference k, or (ii) a new vertex label adjacent to an end vertex of a path can be added or, (iii) two new vertex labels

are added. Our algorithm systematically tries all these possibilities. If none of the cases is possible along a branch, that branch of computation dies. For more details of the algortihm and a proof of correctness, see [8, 9, 10]. The algorithm is shown to be correct in the sense that every graceful labeling of C_n is obtained precisely once. Table 3.3 shows the results for appropriat e values of $n \leq 24$. The last row labeled "Total" shows the total number of graceful labelings for each corresponding value of n in the first row. The other rows show the number of graceful labelings for the corresponding missing label m in the left. Blank cells denote zeros.

n		3	4	7	8	11	12	15	16	19	20	23	24
	0												
	1	1	1										
	2	1		3	3								
	3		1	3	6	26	26						
	4			3	6	42	80	299	299				
	5			3	6	36	80	789	1,476	5,932	5,932		
	6				3	36	120	1,301	3,190	22,210	39,692	162,634	162,634
	7					42	80	1,493	3,494	49,714	104,688	787,218	1,393,740
	8					26	80	1,493	3,646	61,758	162,606	2,218,596	4,813,618
m	9						26	1,301	3,494	72,778	191,238	3,690,788	9,785,048
	10							789	3,190	72,778	196,228	4,633,029	13,567,488
	11							299	1,476	61,758	191,238	5,252,774	15,837,020
	12								299	49,714	162,606	5,253,774	16,280,304
	13									22,210	104,688	4,633,029	15,837,020
	14									5,932	39,692	3,690,788	13,567,488
	15										5,932	2,218,596	9,785,048
	16											787,218	4,813,618
	17											162,634	1,393,740
	18												162,634
	19												
	20												
Total		2	2	12	24	208	492	7,764	20,464	424,784	1,204,540	33,492,078	107,399,400

Table 3.2: Numbers of graceful labelings of C_n for $n \leq 24$

Several properties of graceful labelings of cycles can be gleaned from Table 3.2. We mention two of these below. For proofs, see [9, 10].

Theorem 3.4. $\lceil \frac{n}{4} \rceil \leq m \leq \lfloor \frac{3n}{4} \rfloor$.

Theorem 3.5. *Let $n = 4t$. For the missing label $m = t$, the number of graceful labelings of C_n is equal to the number of graceful labelings of C_{n-1}.*

Some of these results have been generalized to some other classes of unicyclic graphs. See [8, 9, 10] for more details. Truszczyński [25] conjectured in 1984 that all unicyclic graphs, except cycles C_n with $n \equiv 1(mod\ 4)$ or $n \equiv 2(mod\ 4)$, are graceful. Doma [12] studied unicyclic graphs in which the cycle length is 3 through 9, and showed that all unicyclic graphs on up to nine edges (except C_5, C_6 and ,C_9) are graceful. Barrientos [11] studied graceful labelings of a special class of unicyclic graphs. He defined a *hairy cycle* to be a unicyclic graph in which the deletion of any edge in the cycle results in a caterpillar. Barrientos [11] showed that all hairy cycles are graceful.

4. Complexity and Graph Embeddings

Acharya et al [2, 4] proved several results about graceful graph embeddings.

Definition 4.1. *Let G be a graph with graceful labeling f and let $r = m - n + 1$ be the cycle rank of G. Then the labeled graph $G + \overline{K}_r$ where the vertices of $\overline{K}_r$ are assigned the labels of the set $\{0, 1, \cdots, m\} - \{f(v) | v \in V\}$, is called the full augmentation of G and is denoted by G_f.*

Theorem 4.2. [2] *If G is a graceful graph with graceful labeling f, then the graph $H = G_f + \overline{K}_s$ is also graceful for any $s \geq 1$.*

Proof. We assign to the vertices of K_s labels $m + i(m + 1)$ for $1 \leq i \leq s$. It can be easily checked that this gives a graceful labeling of H.   □

Definition 4.3. *Let G be a graph without isolated vertices. The index of gracefulness of G, denoted by $\theta(G)$, is the smallest positive integer k for which it is possible to label the vertices of G with distinct elements from the set $\{0, 1, 2, \ldots, k\}$ in such a way that distinct edges receive distinct labels.*

Such vertex labelings always exist [1, 3]. In fact, for any graph G of order n and size q and with no isolated vertices, it is well known that $\theta(G) \sim O(n^2)$ (cf. P. Erdös in [16]) and G is graceful if and only if $\theta(G) = q$. Thus $\theta(G)$ is a measure of how close G is to being graceful. Given a labeling $f : V(G) \rightarrow \{0, 1, 2, \ldots, \theta(G)\}$, such that the edges of G receive distinct labels, it is easily seen that some vertex of G must be labeled $\theta(G)$, but it is not known whether an edge of G must receive the label $\theta(G)$.

Theorem 4.4. [2] *Any graph G can be embedded as an induced subgraph of a graceful graph.*

Proof. Let $f : V \rightarrow \{0, 1, 2, \ldots, k\}$ be a labeling of G such that the induced function $g_f : E(G) \rightarrow N$ is also injective, where $k = \theta(G)$. Since $k = \theta(G)$, it follows that there exist vertices $u, v \in V$ such that $f(v) = 0$ and $f(u) = k$. Now, let $\{i_1, i_2, \ldots, i_r\}$ be the set of missing edge labels. We assume, without loss of generality, that $i_1, i_2, \ldots, i_s$ are not vertex labels and $i_{s+1}, \ldots, i_r$ are vertex labels. For each i_j, $1 \leq i \leq s$, we add a vertex v_j, join v_j and v by an edge $v_j v$ and define $f(v_j) = i_j$, so that $g_f(v_j v) = i_j$. Hence $i_j, 1 \leq j \leq s$, are edge labels. Now we add new vertices v_j, for each j with $s + 1 \leq j \leq r$, join each v_j to u and v and define $f(v_j) = k + i_j$. Then $g_f(vv_j) = k + i_j$ and $g_f(uv_j) = i_j$. We observe that at this stage $i_1, i_2, \ldots, i_s$ are edge labels and none of the new missing edge labels are vertex labels. For each of these new missing edge labels t, we add a vertex v_t, join v_t and v by an edge and define $f(v_t) = t$. The resulting graph H is graceful and G is an induced subgraph of H.   □

Corollary 4.5. *The problem of deciding whether the chromatic number χ is less than or equal to k, where $k \geq 3$ is NP-complete, even for graceful graphs.*

Proof. Let G be a graph with $\chi(G) \geq 3$. Let H be the graceful graph constructed in Theorem 4.4, which contains G as an induced subgraph. Since all the vertices of $V(H) - V(G)$ are adjacent to either u or v and is not adjacent to any other vertex of G, we have $\chi(H) = \chi(G)$.

Since the problem of deciding whether the chromatic number χ is less than or equal to k, where $k \geq 3$ is NP-complete, ([17], page 190) it follows that the problem of deciding whether the chromatic number χ is less than or equal to k, where $k \geq 3$ is NP-complete even for graceful graphs. $\square$

Acharya et al [4] also proved the following complexity results. These follow from the discussions above.

Theorem 4.6. [4] *The problem of deciding whether the domination number (total domination number) is less than or equal to k is NP-complete even when restricted to graceful graphs.*

Theorem 4.7. [4] *The problem of deciding whether the clique number $\omega(G)$ is greater than or equal to k is NP-complete even when restricted to graceful graphs.*

5. Conclusion

In this paper we have described algorithmic results and complexity issues related to graceful labeling problems. Efforts at proving the Ringel-Kotzig graceful tree conjecture have led to the discovery of several new classes of graceful trees. Generation and enumeration of graceful labelings is a relatively recent area of research and much remains to be done. Open problems in this area include the determination of better bounds for the number of graceful labelings of paths and cycles. It would also be of interest to investigate the index of gracefulness of several classes of graphs. Another interesting problem for further investigation is to find graph theoretic decision problems which remain NP-complete when restricted to the family of graceful graphs. Of course one can look at such problems for graphs which admit other types of labelings as well.

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