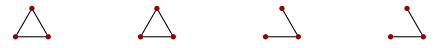
## HW 3

## Due: Fri, 10 Feb 2023

- 1. **Problem 1.3.8.** (-) Which of the following are graphic sequences? Provide a construction item or a proof of impossibility for each.
- 2. Problem 1.3.12. (!) Prove that an even graph has no cut-edge. For each  $k \ge 1$ , construct a (2k + 1)-regular simple graph having a cut-edge. The solution manual gives 3 proofs: (contradiction), (construction/extremality), (prior results), and 3 different constructions. See if you can generate more than one of these. But one is sufficient!
- 3. Problem 1.3.17. (!) Let G be a graph with at least two vertices. Prove or disprove:
  - (a) Deleting a vertex of degree  $\Delta(G)$  cannot increase the average degree.
  - (b) Deleting a vertex of degree  $\delta(G)$  cannot reduce the average degree.
- 4. **Problem 1.3.20.** (!) Count the cycles of length n in  $K_n$ , and the cycles of length 2n in  $K_{n,n}$ . Easy to get wrong if you are not careful.
- 5. Problem 1.3.25. (!) Prove that every cycle of length 2r in a hypercube is contained in a subcube of dimension at most r. Can a cycle of length 2r be contained in a subcube of dimension less than r?
- 6. Problem 1.3.36. Let G be a 4-vertex graph whose list of subgraphs obtained by deleting one vertex appears below. Determine G.



7. Problem 1.3.37. Let H be a graph formed by deleting a vertex from a loopless regular graph G with  $n(G) \ge 3$ . Describe (and justify) a method for obtaining G from H.

## Bonus problems:

- 8. Problem 1.3.26. (!) Count the 6-cycles in  $Q_3$ . Prove that every 6-cycle in  $Q_k$  lies in exactly one 3-dimensional subcube. Use this to count the 6-cycles in  $Q_k$  for  $k \ge 3$ .
- 9. Problem 1.2.36. (+) Alternative characterization of Eulerian graphs.
  - (a) Prove that if G is Eulerian and G' = G uv, then G' has an odd number of u, u-trails that visit v only at the end. Prove also that the number of the trails in this list that are not paths is even. (Toida [1973])

- (b) Let u be a vertex of odd degree in a graph. For each edge e incident to v, let c(e) be the number of cycles containing e. Use  $\sum_{e} c(e)$  to prove that c(e) is even for some e incident to v. (McKee [1984])
- (c) Use part (a) and part (b) to conclude that a nontrivial connected graph is Eulerian if and only if every edge belongs to an odd number of cycles.
- 10. Problem 1.3.16. (+) For  $k \ge 2$  and  $g \ge 2$ , prove that there exists a k-regular graph with girth g. (Hint: To construct such a graph inductively, make use of a (k 1)-regular graph H with girth g and a graph with girth  $\lceil g/2 \rceil$  that is n(H)-regular. (Comment: Such a graph with minimum order is a (k, g)-cage.) (Erdös-Sachs [1963])

See https://en.wikipedia.org/wiki/Cage\_(graph\_theory).