HW 3

Due: Fri, 10 Feb 2023

1. **Problem 1.3.8.** (−) Which of the following are graphic sequences? Provide a construction item or a proof of impossibility for each.

   (a) (5,5,4,3,2,2,2,1),
   (b) (5,5,4,4,2,2,1,1),
   (c) (5,5,5,3,2,2,1, 1),
   (d) (5,5,5,4,2,1,1,1).

2. **Problem 1.3.12.** (!) Prove that an even graph has no cut-edge. For each $k \geq 1$, construct a $(2k+1)$-regular simple graph having a cut-edge.
   The solution manual gives 3 proofs: (contradiction), (construction/extremality), (prior results), and 3 different constructions. See if you can generate more than one of these. But one is sufficient!

3. **Problem 1.3.17.** (!) Let G be a graph with at least two vertices. Prove or disprove:

   (a) Deleting a vertex of degree $\Delta(G)$ cannot increase the average degree.
   (b) Deleting a vertex of degree $\delta(G)$ cannot reduce the average degree.

4. **Problem 1.3.20.** (!) Count the cycles of length n in K_n, and the cycles of length $2n$ in $K_{n,n}$.
   Easy to get wrong if you are not careful.

5. **Problem 1.3.25.** (!) Prove that every cycle of length $2r$ in a hypercube is contained in a subcube of dimension at most r. Can a cycle of length $2r$ be contained in a subcube of dimension less than r?

6. **Problem 1.3.36.** Let G be a 4-vertex graph whose list of subgraphs obtained by deleting one vertex appears below. Determine G.

7. **Problem 1.3.37.** Let H be a graph formed by deleting a vertex from a loopless regular graph G with $n(G) \geq 3$. Describe (and justify) a method for obtaining G from H.

Bonus problems:

8. **Problem 1.3.26.** (!) Count the 6-cycles in Q_3. Prove that every 6-cycle in Q_k lies in exactly one 3-dimensional subcube. Use this to count the 6-cycles in Q_k for $k \geq 3$.

9. **Problem 1.2.36.** (+) Alternative characterization of Eulerian graphs.

   (a) Prove that if G is Eulerian and $G' = G - uv$, then G' has an odd number of u,u-trails that visit v only at the end. Prove also that the number of the trails in this list that are not paths is even. (Toida [1973])

(b) Let u be a vertex of odd degree in a graph. For each edge e incident to v, let $c(e)$ be the number of cycles containing e. Use $\sum_e c(e)$ to prove that $c(e)$ is even for some e incident to v. (McKee [1984])

(c) Use part (a) and part (b) to conclude that a nontrivial connected graph is Eulerian if and only if every edge belongs to an odd number of cycles.

10. **Problem 1.3.16.** (+) For $k \geq 2$ and $g \geq 2$, prove that there exists a k-regular graph with girth g. (Hint: To construct such a graph inductively, make use of a $(k-1)$-regular graph H with girth g and a graph with girth $\lceil g/2 \rceil$ that is $n(H)$-regular. (Comment: Such a graph with minimum order is a (k, g)-cage.) (Erdös-Sachs [1963])
See https://en.wikipedia.org/wiki/Cage_(graph_theory).