HW 5

Due: Fri, 24 Feb 2023

1. **Problem 1.3.8.** Redo part (a) of this one from HW 3, but generate the graph using the intermediate steps from the Havel-Hakimi algorithm. Of course, show your steps.

2. **Problem 2.1.49.** Let G be a simple graph. Prove that $\text{rad}(G) \geq 3 = \text{rad}(\overline{G}) \leq 2$.

3. **Problem 2.2.1.** (–) Determine which trees have Prüfer codes that
   (a) contain only one value,
   (b) contain exactly two values, or
   (c) have distinct values in all positions.

4. **Problem 2.2.24.** Of the ~~n^{n-2} trees~~ *graphs* with vertex set $\{0, .., n-1\}$ *that have* $n-1$ *edges*, how many are gracefully labeled by their vertex names? *Correction as suggested by West in the Soln. Manual.*

5. **Problem 2.3.29.** (–) The game of Scrabble has 100 tiles as listed below. This does not agree with English; "S" is less frequent here, for example, to improve the game. Pretend that these are the relative frequencies in English, and compute a prefix-free code of minimum expected length for transmitting messages. Give the answer by listing the relative frequency for each length of codeword. Compute the expected length of the code (per text character). (Comment: ASCII coding uses five bits per letter; this code will beat that. Of course, ASCII suffers the handicap of including codes for punctuation.)

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	O	R	S	T	U	V	W	X	Y	Z	$\emptyset$
9	2	2	4	12	2	3	2	9	1	1	4	2	6	8	2	1	6	4	6	4	2	2	1	2	1	2

Bonus Problems ....

6. **Problem 2.2.23.** (!) Prove that if the Graceful Tree Conjecture is true and T is a tree with m edges, then K_{2m} decomposes into $2m-1$ copies of T. (Hint: Apply the cyclically invariant decomposition of K_{2m-1} for trees with $m-1$ edges from the proof of Theorem 2.2.16.)
   Note: We did not do Theorem 2.2.16 in class, so you have to read and understand that one first.

7. **Problem 2.3.30.** Consider n messages occurring with probabilities $p_1, \ldots, p_n$, such that each p_i is a power of 1/2 (each $p_i \geq 0$ and $\sum p_i = 1$).
   (a) Prove that the two least likely messages have equal probability.
   (b) Prove that the expected message length of the Huffman code for this distribution is $-\sum p_i \lg p_i$.