HW 6

Due: Fri, 10 Mar 2023

Typo in (5) fixed. Bonus question added.

1. **Problem 5.1.31.** (!) Prove that a graph G is m-colorable if and only if $\alpha(G \,\square\, K_m) \geq n(G)$. (Berge [1973, 379-80])

2. **Problem 5.1.34.** (!) For all $k \in \mathbb{N}$, construct a tree T_k with maximum degree k and an ordering σ of $V(T_k)$ such that greedy coloring relative to the ordering σ uses $k+1$ colors. (Hint: Use induction and construct the tree and ordering simultaneously. Comment: This result shows that the performance ratio of greedy coloring to optimal coloring can be as bad as $(\Delta(G)+1)/2$. (Bean [1976])

3. **Problem 5.2.7.** (!) Given an optimal coloring of a k-chromatic graph, prove that for each color i there is a vertex with color i that is adjacent to vertices of the other $k-1$ colors.

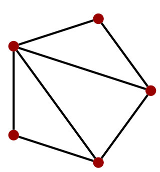
4. **Problem 5.2.15.** (!) Prove that a triangle-free graph with n vertices is colorable with $2\sqrt{n}$ colors. (Comment: Thus every k-chromatic triangle-free graph has at least $k^2/4$ vertices.)

5. **Problem 5.3.4.** Notation: $\chi(G;k) = \chi_G(k)$

   (a) Prove that $\chi(C_n;k) = (k-1)^n + (-1)^n(k-1)$.

   (b) For $H = G \vee K_1$, prove that $\chi(H;k) = k \cdot \chi(G;k-1)$, i.e., $\chi_H(k) = k \cdot \chi_G(k-1)$. From this and part (a), find the chromatic polynomial of the wheel $C_n \vee K_1$.

6. Verify the graph G has chromatic polynomial $\chi_G(k) = k^5 - 7k^4 + 18k^3 - 20k^2 + 8k$.

$G =$ 

Problem 5.3.12. (+) Coefficients of $\chi(G;k)$.

Skip this one, but recall that the leading term of $\chi_G(k)$ is k^n, so the coefficient is 1, and that the second term is $-|E_G|\,k^{n-1}$ (proved in your text). Compare these to part (a) below. Also, there was a question in class about roots of $\chi_G(k)$. See this random paper: Algebraic properties of chromatic roots in https://www.cs.tufts.edu/comp/150GT/documents/ for some info. I was struck by Theorem 2: the location of chromatic roots in $\mathbb{C}$ is related to the theory of phase transitions. (!)

(a) Prove that the last nonzero term in the chromatic polynomial of G is the term whose exponent is the number of components of G.

(b) Use part (a) to prove that if $p(k) = k^n - ak^{n-1} + ... \pm ck^r$ and $a > \binom{n-r+1}{2}$, then p is not a chromatic polynomial. (For example, this immediately implies that the polynomial in Exercise 5.3.3 is not a chromatic polynomial.)

Bonus question

7. Explain where the formula $\chi_{C_n}(k) = (k-1)^n + (-1)^n(k-1)$ comes from. (See **Problem 5.3.4** above.)