COMP 150PP: *Deriving a Density Calculator*
Revised and updated

November 16, 2016

Likelihoods and probabilities

Section 7.1 describes two different hypotheses, each with its own *likelihood*: the density of a distribution over observations.

1. According to generative story e_1 and likelihood function d_1, what is the likelihood of observing $t = 1$?
2. According to generative story e_1 and likelihood function d_1, approximately what is the *probability* of observing $0.9 \leq t \leq 1.1$?
3. According to generative story e_1 and likelihood function d_1, approximately what is the *probability* of observing $1 - \epsilon \leq t \leq 1 + \epsilon$?
4. According to generative story e_1 and likelihood function d_1, approximately what is the *probability* of observing $t = 1$?

Using densities over continuous variables

A laser rangefinder has both a *precision* and an *accuracy*.

- The *precision* estimates fluctuations between measurements of a single distance. If the rangefinder is said to be precise to 1cm, then probably about 2/3 of the time, the measurement is precise within 1cm of the true distance.
- The *accuracy* estimates systematic bias in the rangefinder. For example, if, on average, the rangefinder reports a distance that is 3cm shorter than the true distance is accurate only to within 3cm.

The classic way of quantifying precision and accuracy involves adding normally distributed noise to the true distance. A normal distribution over x has two parameters, x_0 and σ, and it has a density of

$$\frac{1}{\sqrt{2 \cdot \sigma^2 \cdot \pi}} e^{-(x-x_0)^2/2\cdot\sigma^2}$$

5. Given a series of rangefinder observations of a national-standard 100m distance, how would you calculate the maximum likelihood estimate of the rangefinder's precision σ and accuracy x_0?

Probability distributions and densities

In language *Expr a*, the term `Lit 1` denotes a probability distribution.

6. According to the generative story `Lit 1`, what is the likelihood of observing $t = 1$?
7. According to generative story `Lit 1`, approximately what is the *probability* of observing $0.9 \leq t \leq 1.1$?
8. According to generative story `Lit 1`, approximately what is the *probability* of observing $1 - \epsilon \leq t \leq 11 + \epsilon$?
9. According to generative story `Lit 1`, approximately what is the *probability* of observing $t = 1$?
10. Does `Lit 1` have a density? If so, what is it? If not, why not?
11. If a probability distribution over real numbers has a density, what properties should you expect it to have?

Language design

12. In what ways does the language *Expr a* resemble each of the following languages?
    (a) Church
    (b) Wolfe
    (c) The probability monad
13. Could *Expr a* be translated into the probability monad?
    (a) If so, how would that affect the implementation of `sample`?
    (b) If so, how would that affect the implementation of `expect`?
    (c) If not, why not? Where is the difficulty?
14. The key functions in the paper appear to be *sample*, *expect*, and *density*. How are they related?

Calculus review

To differentiate a function composition,

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

where the prime is the derivative operator. Using this law in reverse gives a *change of variables*; the idea is to change x into t by substituting $t = \varphi(x)$. Then, if I have this right,

$$\int_a^b f(\varphi(x)) \cdot \varphi'(x)\ dx = \int_{\varphi(a)}^{\varphi(b)} f(t)\ dt.$$

Calculating densities

15. In section 5.1, the density of distribution `StdRandom`, which we call `u`, appears not to be calculated. Why not? If you try to calculate it in the style of the other examples, what (if anything) goes wrong?

16. Suppose you have a probability distribution of dice in a bowl. When does a density exist? When can a density be calculated? How?

17. In section 5, what key technique from integral calculus is used to support density calculation? How is it used? Does this technique suggest any other primitive syntactic forms whose densities could easily be calculated?

18. Suppose we extend *Expr a* with the new constructor

    ```
    Inv ::  Expr Real -> Expr Real
    ```

    which computes the (multiplicative) inverse, e.g., $\frac{1}{e}$. What should the density calculator do with this term, and why?