COMP 150PP: *Symbolic Bayesian Inference by Lazy Partial Evaluation*

November 21, 2016

Warmup: Two-dimensional distributions

1. In a uniform distribution over the unit square, all points within the square are equally likely. Using the concepts from last week's paper (the density calculator), give precise conditions for "all points are equally likely."
2. Consider a uniform distribution over the unit circle, in which all points are equally likely. Does this distribution have a density? Why or why not? What does the density look like?

Disintegrations: New wine in old bottles

Let's creep up on disintegration by looking at a familiar problem in the world of probability distributions with finite support. Here's our old friend, the experiment "roll a die, toss coins, and see if there are at least three heads."

```
m =
 do { N <- roll d6
    ; coins <- sequence (replicate N flip)
    ; let nheads = length (filter (==H) coins)
    ; return (N, nheads >= 3)
    }
```

Answer these questions:

3. Term `m` denotes a measure over a product space $\alpha \times \beta$. In this example, what are α and β?
4. Find a disintegration of the measure denoted by `m`. That is, find a measure μ of type $\mathbb{M}\ \alpha$ and a (measurable) function κ of type $\alpha \to \mathbb{M}\ \beta$, such that the measure denoted by `m` is $\mu \otimes \kappa$. As a reminder, you know of three ways to write a measure:
   - *Recommended:* An integrator (or over a discrete space, a "summer"). This is the method we used in class a couple of weeks ago. An integrator is a higher-order function which takes an integrand f and returns the integral of f. The integrator $[\![\mu \otimes \kappa]\!](f) = [\![\mu]\!](\lambda a.[\![\kappa\ a]\!](\lambda b.f(a,b)))$. Assume f returns a real number.
   - *Acceptable:* A term of core Hakaru
   - *Not recommended:* A nonnegative, countably additive function from measurable sets to real numbers

Disintegrating the unit square

In the paper by Shan and Ramsey, Figure 2 shows two ways to disintegrate the unit square.

5. Write each way not as a term in core Hakaru, but as a pair $\mu \otimes \kappa$, where μ is a measure, and κ is a function from a real number to a measure. Write each measure as an integrator.

Uniform distribution over the unit circle

6. Write the uniform distribution over the unit circle using a simple integral and a density function. Integrate over the entire space $\mathbb{R}^2$.
7. Using Figure 2 as inspiration, find two *different* iterated integrals that represent the same measure as in the previous problem: a uniform distribution over the unit circle, in which all points are equally likely.
8. Using your results from the previous question, write two different terms of core Hakaru, each of which represents a uniform distribution over the unit circle.

Solving an inference problem using disintegration

Suppose we start with a prior distribution that is uniform over the unit circle, and we observe that every point (x, y) lies on a ray beginning at the origin and pointing 30 degrees east of north (that is, the ray is rotated 60 degrees counterclockwise from the positive x axis).

9. Choose a suitable disintegration and write a measure that represents the posterior distribution. Write an integrator or a term of core Hakaru.
10. Show a definite integral that gives the expected value of x.
11. Pick a different disintegration, and without working through the entire problem a second time, explain how the expected value of x would change.

Extending disintegration to discrete spaces

12. Revisit the term `m` that denotes the result of the experiment "roll a die, toss coins, and see if there are at least three heads." Using the model and examples given in the paper, recast the term into a form in which the *number of heads* is observed and we wish to infer a posterior distribution over N.

13. Write a new solution to the previous problem in which we don't get to observe the number of heads; we get to observe only whether there are at least three heads. Again, write a suitable term of core Hakaru.

14. In the previous problem, what is the product space?

15. Now that you know the product space, write a useful disintegration for problem 13. If there is any real number you don't know, assume it is given to you. Write the disintegration in any form you want.

16. Take your disintegration from the previous problem and write it as a term of core Hakaru.

17. In the paper, equation (4) sets up an observation of a set of *nonzero* measure. Reframe this problem using a disintegration of the product space $\texttt{Bool} \times \mathbb{R}^2$.