CS-151 Quantum Computer Science: Problem Set 6

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Guidelines: ***The deadline to return this problem set is 11.59pm on Tuesday, March 12***. *Remember that you can collaborate with each other in the preliminary stages of your progress, but each of you must write their solutions independently. Submission of the problem set should be via Gradescope only. You are only required to solve one of the questions, and the other requires extra credit (but you should be able to solve both as candidate questions for the exam). Best wishes!*

Problem 1. *We define the* ***quantum Fourier transform*** *on an n-qubit quantum system by its action on the $N = 2^n$ computational basis states $|0\rangle, \cdots, |N-1\rangle$ as,*

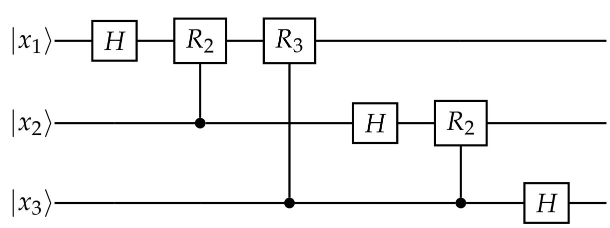
$$QFT : |j\rangle \mapsto \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{2\pi ijk/N} |k\rangle$$

a) *Show that QFT is a unitary transformation.*

b) *In class we mentioned that the Hadamard gate H is the QFT on a 1-qubit ($N = 2^1 = 2$) state. Verify that this is the case, by computing $QFT|0\rangle$ and $QFT|1\rangle$ and comparing it with the action of the Hadamard.*

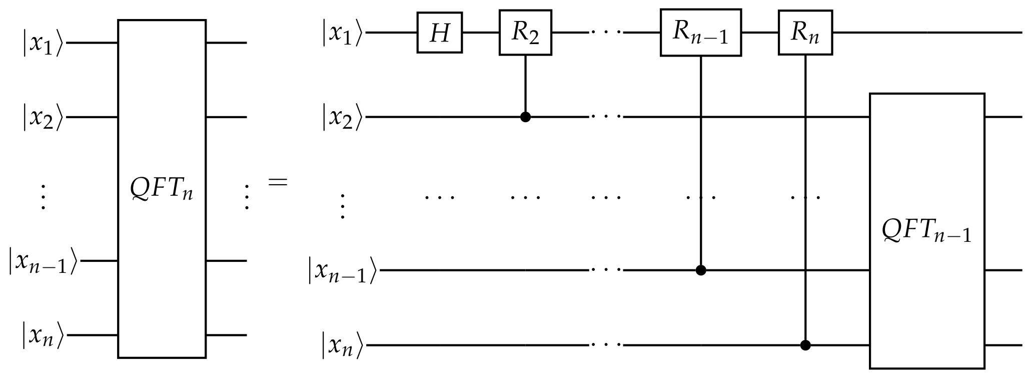
c) *Calculate the (8×8) matrix representation of the the QFT unitary for a 3-qubit system. i.e., $n = 3, N = 2^3 = 8$.*

d) *Let $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ and $R_n := \begin{pmatrix} 1 & 0 \\ 0 & e^{i2\pi/N} \end{pmatrix}$ where $N = 2^n$. Consider the following quantum circuit on $n = 3$ qubits, using only Hadamard H and controlled-phase $C(R_n)$ gates,*



Show that this circuit implements the Quantum Fourier Transform on $n = 3$ qubits.

e) *(Extra credit) Show by induction that the Quantum Fourier Transform satisfies the following recursive relationship between circuit implementations:*



Problem 2 (Order finding using QFT)**.** *Let $N = Lr$ for integers, N, L, r and define,*

$$|\psi\rangle = \frac{1}{\sqrt{L}} \sum_{j=0}^{L-1} |j \cdot r\rangle$$

a) *Apply the $N \times N$ quantum Fourier transform (for $\omega = e^{2\pi i/N}$) on $|\psi\rangle$ and show that the resulting state is given by,*

$$\frac{1}{\sqrt{r}} \sum_{j=0}^{r-1} |jN/r\rangle$$

(Hint: As we saw in class, for any integer T, if $\nu \neq 1$ satisfies $\nu^T = 1$, then $1 + \nu + \nu^2 + \ldots + \nu^{T-1} = 0$)

b) *Instead of $|\psi\rangle$ start with the state,*

$$|\psi'\rangle = \frac{1}{\sqrt{L}} \sum_{j=0}^{L-1} |j \cdot r + x_0 \mod N\rangle$$

for $x_0 \in \{0, 1, \ldots, r-1\}$, and apply the $N \times N$ quantum Fourier transform with $\omega = e^{\frac{2\pi i}{N}}$ to it and measure in the computational basis. Compute the probability of obtaining a specific number $k \in \{0, 1, \ldots, N-1\}$.

c) *Suppose $N = 119$ and upon measuring the above state we obtained 51. What is the value of r?*

d) *Suppose r does not divide N and*

$$|\psi'\rangle = \frac{1}{\sqrt{Q}} \sum_{j=0}^{Q-1} |j \cdot r + x_0 \mod N\rangle$$

and we apply $|\psi''\rangle = QFT|\psi'\rangle$ (Q is the number of terms distinct terms in the above superposition). Write down an expression for the probability of sampling the number $k \in \{0, 1, \ldots, N-1\}$ from the above distribution. Simplify any summations in your expression. (Hint: Use this identity from the class $1 + \nu + \ldots + \nu^{T-1} = \frac{\nu^T - 1}{\nu - 1}$)

e) *Briefly justify why the above probability is maximized when kr is a multiple of N.*