CS 151 Quantum Computer Science

Lecture 6: Single qubit systems

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1 Overview

In the previous lecture, we defined the standard model of quantum computation. We gave an overview of some of the basic quantum gates and introduced the notion of universality for quantum gate-sets. In this lecture, we study single-qubit quantum gates in more detail. After that, we discuss multi-qubit states and introduce "quantum entanglement."

2 Single-qubit rotations:

Recall that the state of a single two-level system is given by a vector in $\mathbb{C}^2$. So we can represent it by:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle, |\alpha|^2 + |\beta|^2 = 1$$

we can use the following angle parameters to representation this qubit

$$|\psi\rangle = \cos(\frac{\theta}{2})|0\rangle + e^{i\phi}\sin(\frac{\theta}{2})|1\rangle$$

The following operator induces a rotation in θ (around Y axis).

$$R_y(\theta) = \begin{pmatrix} \cos(\frac{\theta}{2}) & -\sin(\frac{\theta}{2}) \\ \sin(\frac{\theta}{2}) & \cos(\frac{\theta}{2}) \end{pmatrix}$$

the following creates a rotation around Z

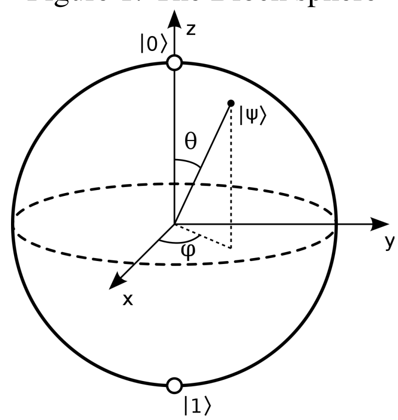
$$R_z(\phi) = \begin{pmatrix} e^{-i\phi/2} & 0 \\ 0 & e^{i\phi/2} \end{pmatrix}$$

the following creates a rotation around X axis.

$$R_x(\theta) = \begin{pmatrix} \cos(\frac{\theta}{2}) & -i\sin(\frac{\theta}{2}) \\ -i\sin(\frac{\theta}{2}) & \cos(\frac{\theta}{2}) \end{pmatrix}$$

We can show that $R_i(\theta)$ performs a rotation by angle θ around the i-th axis. Let us study these operators a bit deeper. To generate θ rotation around X, (Y or Z) axis, we can apply the operator $e^{-i\frac{\theta}{2}Z}$ (same for X and Y). Here for an operator A, e^A is given by the Taylor series

Figure 1: The Bloch sphere



$e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \ldots$. Suppose A has spectral decomposition $A = \sum_i \lambda_i |i\rangle\langle i|$, then $f(A) = \sum_i f(\lambda_i)|i\rangle\langle i|$. In particular

$$Z = |0\rangle\langle 0| - |1\rangle\langle 1|$$

therefore

$$\begin{aligned} e^{-i\frac{\theta}{2}Z} &= e^{-i\frac{\theta}{2}}|0\rangle\langle 0| + e^{i\frac{\theta}{2}}|1\rangle\langle 1| && (1)\\ &= \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix} && (2) \end{aligned}$$

To derive the formula for $R_x(\theta)$, we use the formula $HZH = X$. Therefore for any integer l, $(HZH)^l = X^l$. Plugging these into the Taylor series we can show $e^{-i\frac{\theta}{2}X} = He^{-i\frac{\theta}{2}Z}H$. Therefore

$$\begin{aligned} R_x(\theta) &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} e^{-i\frac{\theta}{2}} & 0 \\ 0 & e^{i\frac{\theta}{2}} \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} && (3)\\ &= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} e^{-i\frac{\theta}{2}} & e^{-i\frac{\theta}{2}} \\ e^{i\frac{\theta}{2}} & -e^{i\frac{\theta}{2}} \end{pmatrix} && (4)\\ &= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} e^{-i\frac{\theta}{2}} & e^{-i\frac{\theta}{2}} \\ e^{i\frac{\theta}{2}} & -e^{i\frac{\theta}{2}} \end{pmatrix} && (5)\\ &= \begin{pmatrix} \cos(\frac{\theta}{2}) & -i\sin(\frac{\theta}{2}) \\ -i\sin(\frac{\theta}{2}) & \cos(\frac{\theta}{2}) \end{pmatrix}. && (6) \end{aligned}$$

Exercise: Prove that

$$e^{-i\frac{\theta}{2}Y} = \begin{pmatrix} \cos(\frac{\theta}{2}) & -\sin(\frac{\theta}{2}) \\ \sin(\frac{\theta}{2}) & \cos(\frac{\theta}{2}) \end{pmatrix} = R_y(\theta)$$

(Hint: $Y = SXS^{-1}$)

How can we describe rotation around an arbitrary direction $\hat{n} = (n_x, n_y, n_z)$ on the Bloch sphere? Let $\sigma = (X, Y, Z)$ and the inner product $\hat{n} \cdot \sigma = n_x X + n_y Y + n_z Z$. When $\hat{n} = (1, 0, 0)$ we obtain X, for $\hat{n} = (0, 1, 0)$ we obtain Y and for $\hat{n} = (0, 0, 1)$ we obtain Z. It turns out rotation by θ around $\hat{n}$ axis is given by $R_{\hat{n}}(\theta) = e^{-i\frac{\theta}{2}\hat{n}\cdot\sigma}$. In general, an arbitrary single-qubit quantum operation can be capture in the form $e^{i\alpha}R_{\hat{n}}(\theta)$ for some unit vector $\hat{n}$.

Exercise: Show that $(\hat{n}\cdot\sigma)^2 = I$. Using this observation show that $R_{\hat{n}} = \cos(\frac{\theta}{2})I + i\sin(\theta)(\hat{n}\cdot\sigma)$. We recommend reading Section 4.2 of Nielsen Chuang for more information.