Reconfiguration of Polygonal Subdivisions via Recombination

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1 Introduction

A redistricting plan is a partition of a geographic domain into connected districts (typically with equal population). A recent trend in redistricting research involves the generation of large ensembles of possible plans using a Markov Chain. The quality of such an ensemble relies on the theoretical ability of the Markov Chain to reach every possible plan from a starting configuration via basic moves.

Motivated by practical considerations such as the resolution of population data, a district map has previously been modeled as a partition of an adjacency graph [1, 2, 3]. In contrast to a discrete setting, we introduce a continuous model for this problem in which the domain is a topological disc.

The most common and successful reconfiguration move is the **recombination**, which is a move that modifies two adjacent districts while maintaining their areas and connectivity [2, 3]. We solve the reachability problem: Given a pair of district maps with the same area distribution, is there a sequence of recombination moves that transforms one into the other? We prove that this is the case for all possible pairs, showing that the configuration space is connected.

2 Reconfiguration Strategy

Every move we make is reversible, so it is enough to show that every configuration can be transformed into a *canonical configuration*. We may assume, due to a homeomorphism, that the domain is a square. In the canonical configuration, the districts are rectangles separated by horizontal lines. See Fig. 1.

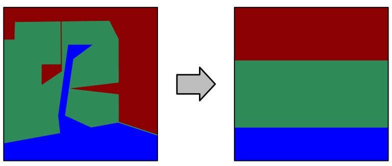

Fig. 1. A starting configuration and the canonical configuration.

We assume that the districts are simple polygons with a total of n vertices. We then reduce the problem with k districts to 3 districts and show that from there, $O(\log n)$ recombination moves are enough to reach the canonical configuration.

Our strategy uses two types of recombination moves: the *gravity move*, where we partition the union of two adjacent districts with a horizontal line, then reconnect disjoint components of each district by infinitesimal "tunnels" along the boundary of the union of the two districts (Fig. 2); and the *exchange move*, where we remove tunnels by exchanging entire portions of a district that are connected through tunnels.

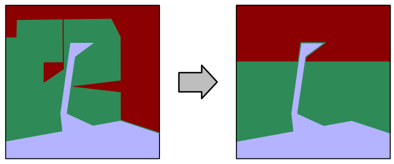

Fig. 2. A gravity move, where we recombine the red and green districts. Since green has two disconnected components after the recombination, we add a tunnel along the boundary that would otherwise be blue-red.

Three Districts. We preprocess the three districts, so that each are topological disks and they are ordered top to bottom—call them T (top), M (middle) and B (bottom). Each of them touches both sides of the bounding square (possibly by a tunnel). From here, we can reach the canonical configuration in $O(\log n)$ moves. First we perform two gravity moves, which removes all of the non-infinitesimal area of T and M from the portion of the map which will belong entirely to B at the end. We then compute a spanning tree of the non-infinitesimal components of the union of T and M, where two components are adjacent if they are connected by a tunnel. Using this tree, we perform $O(\log n)$ *exchange moves* that merge all components by eliminating all tunnels. Then, a final gravity move produces the canonical configuration.

Generalizing. To generalize our strategy, we employ *superdistricts*, that are unions of some adjacent districts. For each move, we recurse on the two superdistricts to solve the single recombination. This leads to the following recursion: $T(k,n) = O(\log n) \cdot T\left(\frac{2k}{3}\right)$, where $T(k,n)$ is the number of moves used to reconfigure a map with k districts and a total of n vertices. This solves to $T(k,n) = k^{O(\log \log n)}$ which simplifies to $O(\log n)$ moves when $k = O(1)$.

References

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