CATEGORY THEORY

INNA ZAKHAREVICH AND LECTURES BY PETER JOHNSTONE

Contents

10/6/06

1. Definitions and Examples

Definition 1.1. A *category* $\mathcal{C}$ consists of:

(i) A collection of *objects* $\mathrm{ob}\,\mathcal{C}$ denoted by $A, B, C, \ldots$
(ii) A collection of *morphisms* $\mathrm{mor}\,\mathcal{C}$ denoted by $f, g, h, \ldots$
(iii) A rule assigning to each $f \in \mathrm{mor}\,\mathcal{C}$ two objects $\mathrm{dom}\, f$ and $\mathrm{cod}\, f$, its *domain* and *codomain*. We write $f : \mathrm{dom}\, f \to \mathrm{cod}\, f$ or $\mathrm{dom}\, f \xrightarrow{f} \mathrm{cod}\, f$.
(iv) For each pair (f, g) of morphisms with $\mathrm{cod}\, f = \mathrm{dom}\, f$ we have a composite morphism $gf : \mathrm{dom}\, f \to \mathrm{cod}\, g$ subject to the axiom $h(gf) = (hg)f$ whenever gf and hg are defined.
(v) For each object A we have an identity morphism $1_A : A \to A$, subject to the axioms $1_B f = f = f 1_A$ for all $f : A \to B$.

Remark. (i) The definition does not depend on any model of set theory. If $\mathrm{ob}\,\mathcal{C}$ is a set then the category is called a *small* category.
(ii) We could eliminate $\mathrm{ob}\,\mathcal{C}$ entirely by using the identity morphisms as stand-ins for objects.

Examples 1.2.

(a) The category **Set** of all sets (objects) and functions (morphisms). (Actually, morphisms are triples (B, f, A) where $f : A \to B$ is a function in the set-theoretic sense (of being a subset of $A \times B$).)
(b) Categories **Gp** of groups, **Rng** of rings, $\mathbf{Mod}_R$ of R-modules, etc have sets with algebraic structure as objects, and homomorphisms as morphisms.
(c) The category **Top** of topological spaces and continuous maps, **Met** of metric spaces and Lipschitz maps, **Diff** of differentiable manifolds and smooth maps, etc.
(d) The category **Htpy** has the same objects as **Top**, but morphisms $X \to Y$ are homotopy classes of functions, with composition induced by function composition. More generally, given a category $\mathcal{C}$ and an equivalence relation $\simeq$ on $\mathrm{mor}\,\mathcal{C}$ such that $f \simeq g$ implies $\mathrm{cod}\, f = \mathrm{cod}\, g$, $\mathrm{dom}\, f = \mathrm{dom}\, g$, and if $f \simeq g$ then $fh \simeq gh$ and $hf \simeq hg$ whenever these are defined we can form the quotient of $\mathcal{C}$ by the equivalence relation to form a *quotient category* $\mathcal{C}/\simeq$.
(e) Given a category $\mathcal{C}$ the opposite category $\mathcal{C}^{\mathrm{op}}$ has the domain and codomain operations interchanged (and thus composition is reversed).
(f) A small category with only one object $*$ is a *monoid* (as any two morphisms are composable). Thus any group is a category.
(g) A *groupoid* is a category in which every morphism in an isomorphism. The *fundamental groupoid* $\pi(X)$ of a space X has points of X as objects, and morphisms $x \to y$ are homotopy classes of paths $x \to y$.
(h) A *discrete* category is one whose only morphisms are identities. So a small discrete category is a set. A *preorder* is a category with at most one morphism $A \to B$ for any two objects A, B. Equivalently, it is a collection of objects with a reflexive transitive relation $\leq$ on it. So a poset is a small preorder whose only isomorphisms are identities. An equivalence relation is a category that is both a preorder and a groupoid.

10/9/06

(i) The category **Rel** has sets as objects, but morphisms $A \to B$ are *relations*, i.e. arbitrary subsets of $B \times A$. Composition of $R : A \to B$ with $S : B \to C$ is defined to be

$$S \circ R = \{(c, a) \mid \exists b \in B \ s.t.\ (c, b) \in S,\ (b, a) \in R\}.$$

This category contains **Set** as a subcategory, and also the category **Part** of sets and partial functions.
(j) Let k be a field. The category $\mathbf{Mat}(k)$ has the natural numbers as objects, and morphisms $n \to m$ are $m \times n$ matrices with entries in k. Composition is matrix multiplication.
(k) Given a theory T in some formal algebra, the category $\mathbf{Der}_T$ has forms of the formal language as objects and morphisms $\varphi \to \psi$ are derivations of ψ from φ. Composition is concatenation.

Definition 1.3. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *functor* $F : \mathcal{C} \to \mathcal{D}$ consists of

(i) a mapping $A \mapsto FA : \mathrm{ob}\,\mathcal{C} \to \mathrm{ob}\,\mathcal{D}$

(ii) a mapping $f \mapsto Ff : \operatorname{mor}\mathcal{C} \to \operatorname{mor}\mathcal{D}$

such that $\operatorname{dom} Ff = F(\operatorname{dom} f)$, $\operatorname{cod} Ff = F(\operatorname{cod} f)$, $F(1_A) = 1_{FA}$, and $F(gf) = (Fg)(Ff)$ whenever gf is defined in $\mathcal{C}$.

Examples 1.4.

(a) We have a functor $U : \mathbf{Gp} \to \mathbf{Set}$ sending a group to its underlying set, and a group homomorphism to itself as a function. Similarly, $U : \mathbf{Top} \to \mathbf{Set}$, $U : \mathbf{Rng} \to \mathbf{Gp}$, etc. We call these *forgetful functors*.

(b) There is a functor $F : \mathbf{Set} \to \mathbf{Gp}$ (the *free functor*) sending a set A to the free group FA generated by A, and a function $f : A \to B$ to the unique homomorphism $Ff : FA \to FB$ sending each generator $a \in A$ to $f(a) \in B \in FB$.

(c) We have a functor $P : \mathbf{Set} \to \mathbf{Set}$ sending A to its power set $P(A) = \{A' \,|\, A' \subset A\}$ and $f : A \to B$ to the mapping $PA \to PB$ sending $A' \subset A$ to $\{f(a) \,|\, a \in A'\} \subset B$. But we also have a functor $P^* : \mathbf{Set} \to \mathbf{Set}^{\mathrm{op}}$ (or $\mathbf{Set}^{\mathrm{op}} \to \mathbf{Set}$) defined by $P^*A = PA$ and $P^*f(B') = f^{-1}(B')$. A functor $\mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ or $\mathcal{C} \to \mathcal{D}^{\mathrm{op}}$ is called a *contravariant functor* $\mathcal{C} \to \mathcal{D}$.

(d) We have a functor $D : \mathbf{Mod}_R^{\mathrm{op}} \to \mathbf{Mod}_R$ sending a module over R to its dual space $DV = V^*$ and a linear map $f : V \to W$ to $f^* : W^* \to V^*$.

(e) We write $\mathbf{Cat}$ for the (large) category of all small categories and functions between them. then $\mathcal{C} \mapsto \mathcal{C}^{\mathrm{op}}$ defines a functor $\mathbf{Cat} \to \mathbf{Cat}$ with f^{op} being f. Note that this is a covariant functor.

(f) A functor between monoids is a monoid homomorphism.

(g) A functor f between posets is an order-preserving map. (Since $a \leq b$ implies a morphism $a \to b$ which maps to a morphism $fa \to fb$, so $fa \leq fb$.)

(h) Let G be a group, considered as a category. A functor $F : G \to \mathbf{Set}$ is a set $A = F*$ equipped with an action of G, i.e. a permutation representation of G. Similarly, for any field k a functor $G \to \mathbf{Mod}_R$ is just a k-linear representation of G.

(i) We have functors $\pi_n : \mathbf{Htpy}_* \to \mathbf{Gp}$, sending a pointed space to its n-th homotopy group. Similarly, we have functors $H_n : \mathbf{Htpy} \to \mathbf{Gp}$ sending a space to its n-th homology.

10/11/06

Definition 1.5. Let $\mathcal{C}, \mathcal{D}$ be two categories and $F, G : \mathcal{C} \rightrightarrows \mathcal{D}$ two functors. A *natural transformation* $\alpha : F \to G$ consists of a mapping $A \mapsto \alpha_A$ $\operatorname{ob}\mathcal{C} \to \operatorname{mor}\mathcal{D}$ such that $\alpha_A : FA \to GA$ for all A and

$$\begin{array}{ccc} FA & \xrightarrow{\alpha_A} & GA \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle Gf} \\ FB & \xrightarrow{\alpha_B} & GB \end{array}$$

commutes for any $f : A \to B$ in $\mathcal{C}$.

Note that, given another functor H and another transformation $\beta : G \to H$ we can form the composite $\beta\alpha$ defined by $(\beta\alpha)_A = \beta_A \alpha_A$.

The composition is associative and has identities so we have a category $[\mathcal{C}, \mathcal{D}]$ of functors $\mathcal{C} \to \mathcal{D}$ and natural transformations between them.

Examples 1.6.

(a) Let k be a field. The double dual operator $V \mapsto V^{**}$ defines a covariant functor $\mathbf{Mod}_k \to \mathbf{Mod}_k$. For every V we have a canonical mapping $\alpha_V : V \to V^{**}$ sending $x \in V$ to the mapping $\varphi \mapsto \varphi(x)$. The α_V's are the components of a natural transformation, and $1_{\mathbf{Mod}_k} \to (-1)^{**}$.

If we restrict to the subcategory $\mathbf{fdMod}_k$ of finite dimensional vector spaces then α_V an isomorphism for all V. This implies that α is an isomorphism in $[\mathbf{fdMod}_k, \mathbf{fdMod}_k]$. In general if α is a natural transformation such that α_A is an isomorphism for all A then the $(\alpha_A)^{-1}$ are also the components of a natural transformation.

(b) Let $P : \mathbf{Set} \to \mathbf{Set}$ be the (covariant) power set functor. There is a natural transformation $\eta : 1_{\mathbf{Set}} \to P$ such that $\eta_A : A \to PA$ sends each $a \in A$ to $\{a\}$. If $f : A \to B$ then $\{f(a)\} = Pf(\{a\})$ holds, so η is indeed natural.

(c) Let G, H be groups and $f, g : G \rightrightarrows H$ two homomorphisms. What is a natural transformation $\alpha : f \to g$? It defines an elements $y = \alpha*$ of H such that for any $x \in G$ we have $yf(x) = g(x)y$. So it is a conjugate between f and g.

(d) For any pointed space (X, x) and every $n \geq 1$ there is a canonical mapping $h_n : \pi_n(X, x) \to H_n(X)$ (the Hurewicz homomorphism). This is a natural transformation from $\pi_n : \mathbf{Htpy}_* \to \mathbf{Gp}$ to the composite

$$\mathbf{Htpy}_* \xrightarrow{U} \mathbf{Htpy} \xrightarrow{H_n} \mathbf{AbGp} \hookrightarrow \mathbf{Gp}$$

Definition 1.7. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor.

(i) We say F is *faithful* if given any two objects $A, B \in \mathcal{C}$ and two morphisms $f, g : A \to B$ $Ff = Fg$ implies $f = g$.
(ii) We say F is *full* if given any two objects $A, B \in \mathcal{C}$ every morphisms $g : FA \to FB$ n $\mathcal{D}$ is of the form Ff for some $f : A \to B$ in $\mathcal{C}$.
(iii) We say a subcategory $\mathcal{C}'$ of $\mathcal{C}$ is *full* if the inclusion $\mathcal{C}' \to \mathcal{C}$ is a full functor.

For example, **AbGp** is a full subcategory of **Gp**, which is a full subcategory of the category **Mod** of monoids. **Diff** is a non-full subcategory of **Top**.

Definition 1.8. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. By an *equivalence* between $\mathcal{C}$ and $\mathcal{D}$ we mean a pair of functors $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$ together with natural isomorphisms $\alpha : 1_C \to GF$ and $\beta : FG \to 1_{\mathcal{D}}$. We write $\mathcal{C} \simeq \mathcal{D}$ if there exists an equivalence between $\mathcal{C}$ and $\mathcal{D}$.

Lemma 1.9. *(Assuming the axiom of choice.) A functor $F : \mathcal{C} \to \mathcal{D}$ is part of an equivalence iff it is full, faithful and essentially surjective on objects. (i.e. every $B \in \operatorname{ob} \mathcal{D}$ is isomorphic to some FA).*

Proof. Suppose we are given G, α, β as in 1.8. For any $B \in \operatorname{ob} \mathcal{D}$ we have $B \cong FGB$ so F is essentially surjective. Suppose that we are given f, g in $\mathcal{C}$ with $Ff = Fg$. Then $GFf = GFg$ so $f = \alpha_B^{-1}(GFf)\alpha_A = \alpha_B^{-1}(GFg)\alpha_A = g$. Thus F is faithful.

Now consider $A, A' \in \operatorname{ob} \mathcal{C}$ and $g : FA \to FA'$. $g : FA \to FA'$ in $\mathcal{D}$. Let f be the composite

$$A \xrightarrow{\alpha_A} GFA \xrightarrow{Gg} GFA' \xrightarrow{\alpha_B^{-1}} A'$$

Then $GFf = Gg$, since both morphisms make the diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & A' \\ {\scriptstyle \alpha_A}\downarrow & & \downarrow{\scriptstyle \alpha_{A'}} \\ GFA & \dashrightarrow & GFA' \end{array}$$

commute. But G is faithful since it is part of an equivalence. So $Ff = g$ and therefore F is full.

Conversely, suppose F is full, faithful, and essentially surjective. For each $B \in \operatorname{ob} \mathcal{D}$ pick a pair (A, β_B) such that $A \in \operatorname{ob} \mathcal{C}$ and $\beta_B : FA \to B$ is an isomorphism. Define $GB = A$. Given $g : B \to B'$ we have a composite

$$FGB \xrightarrow{\beta_B} B \xrightarrow{g} B' \xrightarrow{\beta_{B'}^{-1}} FGB'$$

which must be of the form Ff for a unique $f : GB \to GB'$. Define $Gg = f$. It remains to show that F and G form an equivalence of categories.

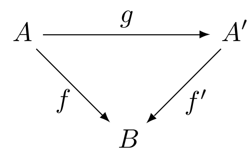
Given $g' : B' \to B''$ the morphisms $(Gg')(Gg)$ and $G(g'g)$ have the same image under F, so they must be equal as F is faithful. Hence G is a functor and β is a natural transformation $FG \to 1_{\mathcal{D}}$. We know $\beta_{FA} : FGFA \to FA$ is an isomorphism, so $(\beta_{FA})^{-1}$ is of the form $F(\alpha_A)$ for a unique $\alpha_A : A \to GFA$ (as F is full) which makes it an isomorphism (as F is faithful). Given $f : A \to A'$ in $\mathcal{C}$ the composites $(\alpha_{A'})f$ and $(GFf)\alpha_A$ have the same image under F by the naturality of β^{-1}, so they are equal. Thus α is a natural transformation $1_{\mathcal{C}} \to GF$ and so we have an equivalence of categories.

10/13/06

∎

Examples 1.10.

(a) Given a category $\mathcal{C}$ and a particular object $B \in \mathcal{C}$ we write $\mathcal{C}/B$ for the category whose objects are morphisms $f : A \to B$ whose morphisms are commutative triangles



and composition induced from composition in $\mathcal{C}$.

For $\mathcal{C} = \mathbf{Set}$ we have an equivalence of categories $\mathbf{Set}/B \cong \mathbf{Set}^B$. The functor $\mathbf{Set}/B \to \mathbf{Set}^B$ sends $f : A \to B$ to $\{f^{-1}(b) \mid b \in B\}$ and $G : \mathbf{Set}^B \to \mathbf{Set}/B$ sends $\{A_b \mid b \in B\}$ to

$$\coprod_{b \in B} A_b = df \cup \{A_b \times \{b\} \mid b \in B\}$$

mapping to B by the second projection.

(b) The 𝔬-slice category $B\backslash\mathcal{C}$ is defined by $(\mathcal{C}^{\mathrm{op}}/B)^{\mathrm{op}}$. In particular $1\backslash\mathbf{Set}$ (where $1 = \{*\}$) is isomorphic to the category $\mathbf{Set}_*$ of pointed sets (via the functor sending $f : 1 \to A$ to $(A, f(*))$). It is also equivalent (but *not* isomorphic) to the category **Part** of sets and partial functions. The functor $F : \mathbf{Set}_* \to \mathbf{Part}$ sends (A, a) to $A\backslash\{a\}$ and $f : (A, a) \to (B, b)$ to the partial f^n which agrees with f at $a \in A$ with $f(a) \neq b$.

In the other direction, $G : \mathbf{Part} \to \mathbf{Set}_*$ sends a set A to $A^+ = A \cup \{A\}$ with A as its base point, and it sends a partial function $f : A \to B$ to f^+ defined by $f^+(a) = f(a)$ if $f(a)$ is defined, and $f^+(a) = B$ otherwise. The composite FG is the identity on **Part**, but GF isn't the identity on **Set**.

Note that in **Part** there is an object $\emptyset$ which is the only member of its isomorphism class, but in $\mathbf{Set}_*$ each isomorphism class contains many members. Hence there can't be an isomorphism of categories between them.

(c) The categories $\mathbf{fdMod}_k$ and $\mathbf{fdMod}_k^{\mathrm{op}}$ are equivalent for any field k via the dual-space functor D and k natural isomorphism $1_{\mathbf{fdMod}_k} \to DD$ (on both sides).

(d) $\mathbf{fdMod}_k$ is also equivalent to $\mathbf{Mat}_k$. To define a functor $F : \mathbf{fdMod}_k \to \mathbf{Mat}_k$ choose a basis for every finite dimensional vector space and define $F(V) = \dim V$, $F(g : V \to W)$ to be the matrix representing G with respect to the chosen bases.

$G : \mathbf{Mat}_k \to \mathbf{fdMod}_k$ sends n to k^n and a matrix A to the linear map represented by A with respect to the standard basis. The composite FG is the identity on $\mathbf{Mat}_k$ (provided we choose the standard basis for k^n for all n). GF isn't the identity but the choice of bases yields a natural isomorphism $GF(V) \to V$ for all V.

Definition 1.11. Given a category $\mathcal{C}$, by a *skeleton* of $\mathcal{C}$ we mean a full subcategory containing exactly one objects from each isomorphism class of objects of $\mathcal{C}$.

Note that lemma 1.9 implies that for any skeleton $\mathcal{C}'$ of $\mathcal{C}$ the inclusion $\mathcal{C}' \to \mathcal{C}$ is part of an equivalence of categories. Also, any equivalence between skeletal categories is bijective on objects, hence is an isomorphism.

Remark. The following statements are each equivalent to the axiom of choice

(i) Any category has a skeleton.
(ii) Any category is equivalent to any of its skeletons.
(iii) Any two skeletons of a given category are isomorphic.

2. The Yoneda Lemma

Definition 2.1. We say a category $\mathcal{C}$ is *locally small* if for any two objects A, B of $\mathcal{C}$ the collection of all morphisms $A \to B$ in $\mathcal{C}$ is a set. We denote this set by $\mathcal{C}(A, B)$.

If $\mathcal{C}$ is locally small then the mapping $B \to \mathcal{C}(A, B)$ becomes a functor $\mathcal{C}(A, -) : \mathcal{C} \to \mathbf{Set}$. Given a morphism $g : B \to C$ in $\mathcal{C}$, $\mathcal{C}(A, g) : \mathcal{C}(A, B) \to \mathcal{C}(A, B)$ sends $f \in \mathcal{C}(A, B)$ to gf. (Associativity of composition implies that this is a functor.) Similarly, $A \mapsto \mathcal{C}(A, B)$ defines a functor $\mathcal{C}(-, B) : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$.

10/16/06

Lemma 2.2 (Yoneda Lemma)**.** (i) *Let $\mathcal{C}$ be a locally small category, $A \in \operatorname{ob}\mathcal{C}$ and $F : \mathcal{C} \to \mathbf{Set}$ a functor. Then there is a bijection between natural transformations $\mathcal{C}(A, -) \to F$ and elements of FA.*
(ii) *Moreover, this bijection is natural in A and F.*

Proof of (i). Given $\alpha : \mathcal{C}(A, -) \to F$ we define $\Phi(\alpha) = \alpha_A(1_A) \in FA$. Conversely, given $x \in FA$ we define $\Psi(x) : \mathcal{C}(A, -) \to F$ by $\Psi(x)_B(f) = (Ff)(x)$ for every $B \in \operatorname{ob}\mathcal{C}$ and $f : A \to B$. We need to verify that $\Psi(x)$ is natural: given $g : B \to C$ we need to check

$$\Psi(x)_C\mathcal{C}(A, g) = (Fg)\Psi(x)_B.$$

But by definition for $f \in \mathcal{C}(A, B)$

$$(Fg)\Psi(x)_B(f) = (Fg)(Ff)(x) = (Fgf)(x) = \Psi(x)_C(gf) = \Psi(x)_C\mathcal{C}(A, g)(f),$$

where the first and third steps are by definition of $\Psi(x)$, the second step is because F is a functor, and the last step is by definition of $\mathcal{C}(A, -)$.

Now we need to check that Ψ and Φ are inverses. Given $x \in FA$ we have $\Phi\Psi(x) = \Psi(x)_A(1_A) = F(1_A)(x) = x$, so $\Phi\Psi$ is the identity. Given any $\alpha : \mathcal{C}(A, -) \to F$ any any $B \in \operatorname{ob}\mathcal{C}$ and $f : A \to B$ we have

$$\alpha_B(f) = \alpha_B(\mathcal{C}(A, f))(1_A) = (Ff)(\alpha_A)(1_A) = (Ff)(\Phi(\alpha)) = (\Psi\Phi(\alpha))_B(f)$$

(where the third step follows by naturality of α), so $\Psi\Phi$ is also the identity and we are done. □

Corollary 2.3. *For a locally small category $\mathcal{C}$ there is a full and faithful functor $Y : \mathcal{C}^{\mathrm{op}} \to [\mathcal{C}, \mathbf{Set}]$ (the Yoneda embedding) sending $A \in \operatorname{ob}\mathcal{C}$ to $\mathcal{C}(A, -)$.*

Proof. Put $F = \mathcal{C}(B, -)$ in Yoneda (i). Hence we have a bijection between morphisms $B \to A$ in $\mathcal{C}$ and morphisms $\mathcal{C}(A, -) \to \mathcal{C}(B, -)$ in $[\mathcal{C}, \mathbf{Set}]$, which we take to be the effect of Y on morphisms. We need to check that this is functorial. Given $C \xrightarrow{g} B \xrightarrow{f} A$ in $\mathcal{C}$. Then $Y(g)Y(f) : \mathcal{C}(A, -) \to \mathcal{C}(C, -)$ is determined by its effect on $1_A \in \mathcal{C}(A, A)$. But $Y(f)_A$ sends 1_A to $f \in \mathcal{C}(B, A)$ and $Y(g)_B(f) = \mathcal{C}(C, f)(g) = fg$, and by definition $Y(fg)_A(1_A) = fg$, so $Y(fg) = Y(f)Y(g)$, as desired. (Note that Y is a contravariant functor.) □

To explain Yoneda (ii), suppose that $\mathcal{C}$ is small. Then $[\mathcal{C}, \mathbf{Set}]$ is locally small, since a natural transformation $F \to G$ is a set-indexed family of functions $\alpha_A : FA \to GA$. We have a functor $\mathcal{C} \times [\mathcal{C}, \mathbf{Set}] \to \mathbf{Set}$ sending (A, F) to FA, and another functor which is the composite

$$\mathcal{C} \times [\mathcal{C}, \mathbf{Set}] \xrightarrow{Y \times 1_{[\mathcal{C}, \mathbf{Set}]}} [\mathcal{C}, \mathbf{Set}]^{\mathrm{op}} \times [\mathcal{C}, \mathbf{Set}] \xrightarrow{[\mathcal{C}, \mathbf{Set}](-,-)} \mathbf{Set}$$

(ii) is saying that these two functors are naturally isomorphic in each variable. Notice, however, that since the existence of a natural isomorphism is a purely "local" condition, we only need to require that the category be locally small.

Proof of (ii). For naturality in A, suppose that we are given $f : A \to B$, a functor F and a natural transformation $\alpha : \mathcal{C}(A, -) \to F$. We need to show that $(Ff)\Phi(\alpha) = \Phi(\alpha \circ Y(f))$. But

$$\Phi(\alpha \circ Y(f)) = \alpha_B(Y(f)_B(1_B)) = \alpha_B(f) = \alpha_B(\mathcal{C}(A, f)(1_A)) = (Ff)(\alpha_A(1_A)) = (Ff)\Phi(\alpha),$$

where the second-to-last step follows by naturality.

For naturality in F, suppose that we are given $\theta : F \to G$ and $\alpha : \mathcal{C}(A, -) \to F$. We need to verify that $\theta_A\Phi(\alpha) = \Phi(\theta \circ \alpha)$ as elements of GA. But both of these are $\theta_A(\alpha_A(1_A))$ by definition, so we are done. □

Definition 2.4. We say that a functor $F : \mathcal{C} \to \mathbf{Set}$ is *representable* if it is naturally isomorphic to $\mathcal{C}(A, -)$ for some A. By a *representation* of F we mean a pair (A, x) where $A \in \operatorname{ob}\mathcal{C}$ and $x \in FA$ is such that $\Psi(x) : \mathcal{C}(A, -) \to F$ is an isomorphism. We call x a *universal element* of F. It has the property that any $y \in FB$ is of the form $(Ff)(x)$ for some $f \in \mathcal{C}(A, B)$.

Corollary 2.5. *Given two representations (A, x) and (B, y) of the same functor F there is a unique isomorphism $f : A \to B$ in $\mathcal{C}$ such that $Ff(x) = y$.*

Proof. Consider the composite

$$\mathcal{C}(B,-) \xrightarrow{\Psi(y)} F \xrightarrow{\Psi(x)^{-1}} \mathcal{C}(A,-)$$

By corollary 2.3 there exists a unique $f \in \mathcal{C}(A,B)$ with $Yf = \Psi(x)^{-1}\Psi(y)$ and a unique $g : B \to A$ with $Yg = (Yf)^{-1}$, with fg and gf being identities (because Y is faithful). Moreover, the equation $Yf = \Psi(x)^{-1}\Psi(y)$ is equivalent to $\Psi(x)Y(f) = \Psi(y)$, but these are equal iff they have the same effect on 1_B, i.e. iff $(Ff)(x) = y$. □

Examples 2.6.

(a) The forgetful functor $U : \mathbf{Gp} \to \mathbf{Set}$ is representable by $(\mathbb{Z}, 1)$ since for any group G and $x \in UG$ there is a unique homomorphism $\mathbb{Z} \to G$ sending 1 to x. Similarly, $U : \mathbf{Top} \to \mathbf{Set}$ is representable by $(\{*\}, *)$.

(b) The contravariant power set functor $P^* : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Set}$ is representable by $(\{0,1\}, 1)$ since for any $A' \subseteq A$ there is a unique $\chi_{A'} : A \to \{0,1\}$ such that $\chi_{A'}^{-1}(1) = A'$.

(c) For a field k the composite functor $\mathbf{Mod}_k^{\mathrm{op}} \xrightarrow{-^*} \mathbf{Mod}_k \xrightarrow{U} \mathbf{Set}$ is representable by $(k, 1_k)$.

10/18/06

(d) Let G be a group. The category $[G, \mathbf{Set}]$ is the category of sets with a G-action. The (unique) representable functor $G \to \mathbf{Set}$ is the *Cayley representation* of G, i.e. G itself with action by left multiplication. In this case the Yoneda Lemma tells us that this is the free G-set on one generator, i.e. that morphisms $G \to A$ in $[G, \mathbf{Set}]$ correspond bijectively to elements of A.

(e) Let $\mathcal{C}$ be a locally small category, A and B two objects of $\mathcal{C}$. Consider the functor $F : \mathcal{C}(-,A) \times \mathcal{C}(-,B) : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$. What does it mean for this to be representable? A representation consists of an object P together with an element $(p : P \to A, q : P \to B)$ of FP, such that for any C and any $f : C \to A$, $g : C \to B$ there is a unique $h : C \to P$ such that $ph = f$ and $qh = g$.

We can ask whether this exists in any category $\mathcal{C}$, not necessarily locally small. If it does, we call (P, p, q) a (categorical) *product* of A and B (and normally denote it by $(A \times B, \pi_1, \pi_2)$).

Note that in $\mathbf{Set}$ it is the usual Cartesian product $A \times B$ equipped with the two projections. Give $f : C \to A$ and $g : C \to B$ we define h by $h(c) = (f(c), g(c))$. In $\mathbf{Gp}$, $\mathbf{Rng}$, $\mathbf{Top}$, etc. the products exist and are constructible by taking the Cartesian product of the underlying sets.

A *coproduct* in $\mathcal{C}$ is a product in $\mathcal{C}^{\mathrm{op}}$; usually denote the coproduct of A and B by $A \amalg B$. In $\mathbf{Set}$ the coproduct of sets is a disjoint union. This also makes sense in $\mathbf{Top}$. In $\mathbf{Gp}$ the coproduct of two group sis their *free product* $G * H$. In $\mathbf{AbGp}$ $G \amalg H = G \times H$ and is usually denoted by $G \oplus H$. In any poset $(P, \leq)$ a product $a \times b$ is a greatest lower bound $(a \wedge b)$ and a coproduct is a least upper bound $(a \vee b)$.

(f) Assume $\mathcal{C}$ is locally small. Suppose we are given a parallel pair $f, g : A \to B$ in $\mathcal{C}$; consider the functor F defined by $F(C) = \{h : C \to A \mid fh = gh\}$ (which is a subfunctor of $\mathcal{C}(-,A)$). Is this representable?

A representation consists of (E, e) where $e : E \to A$ satisfies $fe = ge$ and any $h : C \to A$ with $fh = gh$ factors uniquely as ek for $k : C \to E$. Such an e is called an *equalizer* of f and g.

In $\mathbf{Set}$ we take $E = \{a \in A \mid f(a) = g(a)\}$ and e the inclusion map. This construction also works in $\mathbf{Gp}$, $\mathbf{Rng}$, $\mathbf{Mod}_R$, $\mathbf{Top}$, … The dual notion is that of a *coequalizer*; again it exists in all of the above categories, but the constructions are different.

Definition 2.7. We say a morphism $f : A \to B$ is a *monomorphism* if $fg = fh \Rightarrow g = h$ for all $g, h : C \to A$. Dually, f is an *epimorphism* if $kf = \ell f \Rightarrow k = \ell$ for all $k, \ell : B \to C$. We say f is a *regular monomorphism* if it arises as the equalizer of some pair of maps, and a *regular epimorphism* if it arises as the coequalizer of some pair of maps.

In $\mathbf{Set}$ the monomorphisms are all regular, and are exactly the injective maps. To see this suppose f is injective and consider $C = B \times \{0,1\}/\sim$ where $(b,j) \sim (c,k)$ iff either $b = c$ and $j = k$ or $b = c = f(a)$ for some $a \in A$. Then the two injections $B \rightrightarrows C$ have equalizer $\{b \in B \mid \exists\, a \in A \text{ s.t. } b = f(a)\}$, which means that f is a regular monomorphism. If f is not injective then we can find $x, y : 1 \to A$ such that $x \neq y$ but $f(x) = f(y)$, so f is not a monomorphism.

Similarly we can show that in $\mathbf{Set}$ all epimorphisms are regular and are exactly the surjective maps.

However, these equivalences don't hold in all familiar categories. They hold in **Gp** but not in **Mon**, since the inclusion $\mathcal{N} \to \mathbb{Z}$ is an epimorphism in **Mon**. It's also a monomorphism, but it is not a regular monomorphism, since an epic equalizer has to be an isomorphism. Similarly, in **Top** the monomorphisms are the injective functions and the epimorphisms are the surjective functions, but the regular monomorphisms are only the subspace injections, and the regular epimorphisms are only the quotients by a subspace, as the imposition of a topology makes the regularity condition stronger. Note also that there are bijective continuous maps which aren't homeomorphisms.

We say that a category $\mathcal{C}$ is *balanced* if every morphism which is both epic and monic is an isomorphism. (Thus **Set** and **Gp** are balanced, but **Mod** and **Top** are not.)

Definition 2.8. Let $\mathcal{C}$ be a category, $\mathscr{G}$ a class of objects in $\mathcal{C}$.

(i) We say $\mathscr{G}$ is a *separating family* if, given $f, g : A \to B$ with $f \neq g$ there exists $G \in \mathscr{G}$ and $h : G \to A$ with $fh \neq gh$.
(ii) We say $\mathscr{G}$ is a *detecting family* if given $f : A \to B$ such that every $g : G \to B$ with $G \in \mathscr{G}$ factors uniquely as fh, then f is an isomorphism.

If a category is locally small then $\mathscr{G}$ is a separating family iff $\{\mathcal{C}(G, -) \,|\, G \in \mathscr{G}\}$ is "jointly faithful." $\mathscr{G}$ is a detecting family iff $\{\mathcal{C}(G, -) \,|\, G \in \mathscr{G}\}$ is "jointly isomorphism-reflecting."

10/20/06

Lemma 2.9.

(i) *Suppose $\mathcal{C}$ has equalizers for all parallel pairs. Then every detecting family of objects of $\mathcal{C}$ is a separating family.*
(ii) *Suppose $\mathcal{C}$ is balanced. Then every separating family of objects of $\mathcal{C}$ is a detecting family.*

Proof.

(i) Suppose $\mathscr{G}$ is a detecting family, and suppose $f, g : A \to B$ is such that every $h : G \to A$ with $G \in \mathscr{G}$ satisfies $fh = gh$. Then every such h factors uniquely through the equalizer $e : E \to A$ of (f, g), so e is an isomorphism. Hence $f = g$.
(ii) Suppose $\mathscr{G}$ is a separating family, and suppose $f : A \to B$ is such that any $g : G \to B$ with $G \in \mathscr{G}$ factors uniquely through f. Then f is epic, since if $h, k : B \to C$ satisfies $hf = kf$ then any $g : G \to B$ must satisfy $hg = kg$, so $h = k$. Similarly, if $\ell, m : D \to A$ satisfies $f\ell = fm$ then for any $n : G \to D$ we have $f\ell n = fmn$, so ℓn and mn are both factorizations of $f\ell n$ through f, so they're equal. Hence $\ell = m$, so f is monic. Since $\mathcal{C}$ is balanced, f is an isomorphism.

□

Examples 2.10.

(a) $\mathrm{ob}\,\mathcal{C}$ is always both a detecting and separating family for $\mathcal{C}$. For example, if $f : A \to B$ is such that every $g : C \to B$ factors uniquely through f, then there exists a unique $h : B \to A$ such that $fh = 1_B$. Then hf and 1_A are both factorizations from f through f, so they're equal.
(b) For any locally small $\mathcal{C}$, $\{YA \,|\, A \in \mathrm{ob}\,\mathcal{C}\}$ is a separating and detecting family for $[\mathcal{C}, \mathbf{Set}]$. For if $\alpha : F \to G$ is an arbitrary natural transformation, then if every $YA \to C$ factors uniquely through α, α_A is bijective, and if this holds for all A then α is an isomorphism.
(c) $\{1\}$ is both a separating and a detecting family for **Set**, since $\mathbf{Set}(1, -)$ is isomorphic to an identity functor. $\{\mathbb{Z}\}$ is both for **Gp** (or **AbGp**), since $\mathbf{Gp}(\mathbb{Z}, -)$ is isomorphic to the forgetful functor. $\{\mathbb{Z}\}$ is both for $\mathbf{Set}^{\mathrm{op}}$, since $\mathbf{Set}(-, \mathbb{Z})$ is isomorphic to P^*, which is faithful.
(d) $\{1\}$ is a generating family for **Top**, since $\mathbf{Top} \to \mathbf{Set}$ is faithful. However, **Top** has no detecting *set* of objects: for any infinite cardinal K we can find a set X (of cardinality K) and two topologies $\mathcal{T}_0, \mathcal{T}_1$ of X such that $\mathcal{T}_1 \supsetneq \mathcal{T}_2$ but the two topologies coincide on any subset of X of cardinality less than K. Given any set $\mathscr{G}$ of objects of **Top**, choose $K > \#(UG)$ for any $G \in \mathscr{G}$. Then $\mathscr{G}$ can't detect the fact that $1_X : (x, \mathcal{T}_1) \to (x, \mathcal{T}_2)$ isn't an isomorphism.
(e) Let $\mathcal{C}$ be the category of connected pointed CW-complexes and homotopy classes of continuous maps between them. JHC Whitehead's theorem asserts that if $f : X \to Y$ in this category induces isomorphisms $\pi_n(X) \to \pi_n(Y)$ for all $n \geq 1$ then it is an isomorphism. But $U\pi_n$ (where U is the forgetful functor $\mathbf{Gp} \to \mathbf{Set}$) is represented by S^n, so it says that $\{S^n \,|\, n \geq 1\}$ is a detecting set for

$\mathcal{C}$. However, PJ Freyd showed that there is no faithful functor $\mathcal{C} \to \mathbf{Set}$, hence there is no separating set of objects of $\mathcal{C}$. (If $\mathscr{G}$ were a separating set then $x \mapsto \coprod_{G \in \mathscr{G}} \mathcal{C}(G, X)$ would be faithful.)

Definition 2.11. Let $\mathcal{C}$ be a category, $P \in \operatorname{ob}\mathcal{C}$. We say that P is *projective* if, given any diagram of the form

$$\begin{array}{ccc} & & A \\ & \overset{h}{\nearrow} & \downarrow f \\ P & \xrightarrow{g} & B \end{array}$$

with f epic there exists $h : P \to A$ with $fh = g$. (If $\mathcal{C}$ is locally small, this says that $\mathcal{C}(P, -)$ preserves epimorphisms.) We say that P is *injective* in $\mathcal{C}$ if it is projective in $\mathcal{C}^{\mathrm{op}}$. More generally, if $\mathscr{E}$ is a class of epimorphisms in $\mathcal{C}$ we say P is $\mathscr{E}$-projective if the above holds for all $f \in \mathscr{E}$.

Lemma 2.12. *Let $\mathcal{C}$ be locally small. Then for any $A \in \operatorname{ob}\mathcal{C}$ YA is $\mathscr{E}$-projective in $[\mathcal{C}, \mathbf{Set}]$, where $\mathscr{E}$ is the class of natural transformations α such that α_B is surjective for all B. (In fact, these are all of the epimorphisms in $[\mathcal{C}, \mathbf{Set}]$.)*

Proof. Given $\begin{array}{ccc} & & F \\ & & \downarrow \alpha \\ YA & \xrightarrow{\beta} & G \end{array}$ β corresponds to some $y \in GA$. As α_A is surjective $y = \alpha_A(x)$ for some $x \in FA$. Then $\alpha\Psi(x) = \beta$ so $\Psi(x)$ completes the above diagram. □

3. Adjunctions

Definition 3.1. Suppose we are given categories $\mathcal{C}, \mathcal{D}$ and functors $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$. We say that F is *left adjoint* to G or G is *right adjoint* to F we're given, for each $A \in \operatorname{ob}\mathcal{C}$ and each $B \in \operatorname{ob}\mathcal{D}$ a bijection between morphisms $FA \to B$ in $\mathcal{D}$ and morphisms $A \to GB$ in $\mathcal{C}$, which is natural in A and B. (If $\mathcal{C}$ and $\mathcal{D}$ are locally small this means that the functors $\mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathbf{Set}$ sending (A, B) to $\mathcal{D}(FA, B)$ and to $\mathcal{C}(A, GB)$ are naturally isomorphic.) We write $(F \dashv G)$ if F is left adjoint to G.

Note that the naturality condition means that

$$\begin{array}{ccc} FA & \xrightarrow{h} & B \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle g} \\ FC & \xrightarrow{j} & D \end{array} \quad \text{commutes iff} \quad \begin{array}{ccc} A & \xrightarrow{\widehat{h}} & GB \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle Gg} \\ C & \xrightarrow{\widehat{j}} & GD \end{array} \quad \text{commutes.}$$

10/23/06

Examples 3.2.

(a) The functor $F : \mathbf{Set} \to \mathbf{Gp}$ is left adjoint to the forgetful functor U. For any function $A \to UG$ there is a unique homomorphism $T : FA \to G$ extending f (and this is natural in both A and G). Similarly for free rings, R-modules, etc.

(b) The forgetful functor $U : \mathbf{Top} \to \mathbf{Set}$ has as a left adjoint D, sending any set A to A with the discrete topology (since any function $A \to UX$ is continuous as a map $DA \to X$). U has a right adjoint I, sending A to A with the indiscrete topology $\{A, \emptyset\}$.

(c) The functor $\operatorname{ob} : \mathbf{Cat} \to \mathbf{Set}$ has a left adjoint D sending A to the discrete category whose objects are the members of A. (since a functor $DA \to \mathcal{C}$ is uniquely determined by its effect on objects) and a right adjoint I sending A to the preorder with objects $a \in A$ and one morphism $a \to b$ for all $(a, b) \in A \times A$. (Again, a functor $\mathcal{C} \to IA$ is uniquely determined by its effect on objects.) In this case D also has a left adjoint π_0 sending $\mathcal{C}$ to its set of *connected components*, i.e. equivalences of objects A with $U \sim V$ if there exists a morphism $U \to V$. (Once again, a functor $\mathcal{C} \to DA$ is determined by its effect on objects, but the functor $\operatorname{ob}\mathcal{C} \to A$ has to be ordered on connected components.)

(d) Let 1 denote the category with one object and one morphism. For any $\mathcal{C}$ there's a unique functor $\mathcal{C} \to 1$. A left adjoint (if it exists) picks out an *initial object* of $\mathcal{C}$, i.e. an object $\emptyset$ such that there exists a unique $\emptyset \to A$ for all $A \in \operatorname{ob}\mathcal{C}$. Similarly, a right adjoint picks out a *terminal object* $*$ of $\mathcal{C}$, i.e. one such that there is a unique morphism $A \to *$ for all A.

(e) Let $(X, \mathcal{T})$ be a topological space. If we think of $\mathcal{T}$ as a poset (ordered by inclusion) then $\mathcal{T} \to PX$ is a functor. The operation $A \mapsto A^o$ (the interior of A) is a right adjoint to this functor, since by definition we have $U \subseteq A$ iff $U \subset A^o$ for $U \in \mathcal{T}$. Similarly, closure is a left adjoint to the inclusion of the poset of closed sets in PX.

(f) The functor $P^* : \mathbf{Set} \to \mathbf{Set}^{\mathrm{op}}$ is left adjoint to $P^* : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Set}$, since morphisms $P^*A \to B$ in $\mathbf{Set}^{\mathrm{op}}$ are functions $B \not\to P^*A$ in $\mathbf{Set}$ which correspond to relations $B \to A$ and morphisms $A \to P^*B$ in $\mathbf{Set}$ correspond to relations $A \not\to B$. These correspond bijectively in a natural way. This becomes a symmetric relation and we write it as $\mathbf{Set}(A, P^*B) \cong \mathbf{Set}(A, P^*B)$. We say P^* is *self-adjoint on the right*.

(g) Given two sets A and B and a relation between them $R \subseteq A \times B$ we have a mapping $\cdot^r : PA \to PB$ sending $S \subseteq A$ to $S^r = \{b \in B \mid \forall a \in S,\ (a, b) \in R\}$, and mapping sending $T \subseteq B$ to $T^\ell = \{a \in A \mid \forall b \in T\ (a, b) \in R\}$. These are contravariant functors, adjoint on the right since $T \subseteq S^r$ iff $S \times T \subseteq R$ iff $S \subseteq T^\ell$.

Theorem 3.3. *Suppose we are given $G : \mathcal{D} \to \mathcal{C}$. For each object A of $\mathcal{C}$ consider the category $(A \downarrow G)$ whose objects are pairs (B, f) with $B \in \operatorname{ob}\mathcal{D}$ and $f : A \to GB$ in $\mathcal{C}$, and whose morphisms $(B, f) \to (B', f')$ are morphisms $g : B \to B'$ such that $f' = (Gg)f$. The specifying a left adjoint for G is equivalent to specifying an initial object of $(A \downarrow G)$ for each A.*

Proof. Suppose G has a left adjoint F. For any A the morphism $1 : FA \to FA$ corresponds to a morphism $\eta_A : A \to GFA$, called the *unit* of the adjunction. We claim that (FA, η_A) is an initial object of $(A \downarrow G)$. For, given an arbitrary object (B, f) the diagram

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & GFA \\ & \searrow^{f} & \downarrow{\scriptstyle Gg} \\ & & GB \end{array}$$

commutes iff f is the morphism corresponding to $FA \xrightarrow{1} FA \xrightarrow{g} B$.

10/25/06

Now suppose that we are given an initial object of $(A \downarrow G)$ for each $A \in \operatorname{ob}\mathcal{C}$. Denote this object by (FA, η_A); this defines F on objects. Given $f : A \to A'$ in $\mathcal{C}$, define $Ff : FA \to FA'$ to be the unique morphism such that

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & GFA \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle GFf} \\ A' & \xrightarrow{\eta'_A} & GFA' \end{array}$$

commutes, i.e. the unique morphism $(FA, \eta_A) \to (FA', \eta_{A'} f)$ in $(A \downarrow G)$.

If we have $f' : A' \to A''$ then $F(f'f)$ and $(Ff')(Ff)$ are both morphisms $(FA, \eta_A) \to (FA'', \eta_A f'f)$ so they must be equal: hence F is a functor, and η is a natural transformation $1_{\mathcal{C}} \to GF$. We have a bijective correspondence between morphisms $f : A \to GB$ and morphisms $g : FA \to B$: take g to be the unique morphism such that $(Gg)\eta_A = f$. Naturality in B is immediate from the form of the definition; naturality in A follows from the fact that η is a natural transformation. □

Corollary 3.4. *Any two left adjoints F, F' for a given functor G are (canonically) naturally isomorphic.*

Proof. For each A there's a unique isomorphism $(FA, \eta_A) \to (F'A, \eta'_A)$ in $(A \downarrow G)$; it's easy to verify that this is natural in A. □

Lemma 3.5. *Given functors $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D} \underset{K}{\overset{H}{\rightleftarrows}} \mathcal{E}$ with $(F \dashv G)$ and $(H \dashv K)$, then $(HF \dashv GK)$.*

Proof. We have bijections between morphisms $HFA \to C$, morphisms $FA \to KC$ and morphisms $A \to GKC$ natural in A and C. Compose these to get bijections between $HFA \to C$ and $A \to GKC$ natural in A and C. □

Corollary 3.6. *Suppose we are given a commutative square of categories and functors*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{G_1} & \mathcal{D} \\ {\scriptstyle G_2}\downarrow & & \downarrow{\scriptstyle G_3} \\ \mathcal{E} & \xrightarrow[G_4]{} & \mathcal{F} \end{array}$$

and suppose each G_i has a left adjoint F_i. Then

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{F_4} & \mathcal{E} \\ {\scriptstyle F_3}\downarrow & & \downarrow{\scriptstyle F_2} \\ \mathcal{D} & \xrightarrow[F_1]{} & \mathcal{C} \end{array}$$

commutes up to natural isomorphism.

Given functors $F : \mathcal{C} \to \mathcal{D} : G$ with $(F \dashv G)$ we have a natural transformation $\eta : 1_{\mathcal{C}} \to GF$ and dually a natural transformation $\epsilon : FG \to 1_{\mathcal{D}}$ (the *counit* of the adjunction).

Theorem 3.7. *Given functors $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$, specifying an adjunction $F(\dashv G)$ is equivalent to specifying natural transformations $\eta : 1_{\mathcal{C}} \to GF$ and $\epsilon : FG \to 1_{\mathcal{D}}$ satisfying the triangular identities:*

$$\begin{array}{ccc} F & \xrightarrow{F\eta} & FGF \\ & {\scriptstyle 1_F}\searrow & \downarrow{\scriptstyle \epsilon_F} \\ & & F \end{array} \qquad \textit{and} \qquad \begin{array}{ccc} G & \xrightarrow{\eta_G} & GFG \\ & {\scriptstyle 1_G}\searrow & \downarrow{\scriptstyle G\epsilon} \\ & & G \end{array}$$

Proof. Suppose we are given an adjunction $(F \dashv G)$ with unit η and counit ϵ. By definition $\eta_A : A \to GFA$ corresponds to $1_{FA} : FA \to FA$ and $\epsilon_{FA} : FGFA \to FA$ corresponds to $1_{GFA} : GFA \to GFA$. So $\epsilon_{FA}(F\eta_A) : FA \to FA$ corresponds to $A \xrightarrow{\eta_A} GFA \xrightarrow{1_{GFA}} GFA$. Hence $\epsilon_{FA}(F\eta_A) = 1_{FA}$ as desired. The dual argument shows the statement for the other triangle.

Conversely, suppose we are given η and ϵ satisfying the identities. For any $f : A \to GB$ define $\Phi(f) : FA \to B$ to be the composite $FA \xrightarrow{Ff} FGB \xrightarrow{\epsilon_B} B$. Given $g : FA \to B$ define $\Psi(g) : A \to GB$ to be $A \xrightarrow{\eta_A} GFA \xrightarrow{Gg} GB$. As in the proof of 3.3 we know that Ψ and Φ are natural in A and B. To show that they are inverses to each other,

$$\begin{aligned} \Psi\Phi(f) &= A \xrightarrow{\eta_A} GFA \xrightarrow{G\Phi f} GB \\ &= A \xrightarrow{\eta_A} GFA \xrightarrow{GFf} GFGB \xrightarrow{G\epsilon_B} GB \\ &= A \xrightarrow{f} GB \xrightarrow{\eta_{GB}} GFGB \xrightarrow{G\epsilon_B} GB \\ &= A \xrightarrow{f} GB \end{aligned}$$

where the third line follows because η is natural, and the last one is by the second triangle identity. Similarly, $\Phi\Psi(g) = g$ for all $g : FA \to B$. □

Lemma 3.8. *Suppose that we are given $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$, $(F \dashv G)$ with counit $\epsilon : FG \to 1_{\mathcal{D}}$. Then*

(i) *G is faithful iff ϵ_B is an epimorphism for all B.*
(ii) *G is full and faithful iff ϵ is an isomorphism.*

Proof.

(i) Suppose that ϵ_B is epic for all B, and suppose $g, g' : B \to B'$ satisfy $Gg = Gg'$. Then the morphisms $FGB \to B'$ corresponding Gg and Gg' are equal, but these are $g\epsilon_B$ and $g'\epsilon_B$, respectively. As ϵ_B is epic, $g = g'$.

Conversely, suppose that G is faithful and $g, g' : B \to B'$ satisfy $g\epsilon_B = g'\epsilon_B$. Then $Gg = Gg'$, so $g = g'$.

(ii) Suppose ϵ is an isomorphism. As any isomorphism is epic we know that G is faithful so we only need to show that G is full. Suppose that we are given $f : GB \to GB'$. Transposing, we get $\overline{f} : FGB \to B'$. Then if we set $g = \overline{f}\epsilon_B^{-1} : B \to B'$ we have Gg corresponding to $\overline{f}$, so $Gg = f$.

Conversely, suppose that G is full and faithful. Then $\eta_{GB} : GB \to GFGB$ must be of the form Gh for a unique $h : B \to FGB$; but $(G\epsilon_B)(\eta_{GB}) = 1_{GB}$, so $\epsilon_B h = 1_B$ since G is faithful. $h\epsilon_B$ corresponds under the adjunction to $(Gh)\mathrm{id}_{GB} = \eta_{GB}$, so $h\epsilon_B = 1_{FGB}$.

□

Definition 3.9. By a *reflexion* we mean an adjunction satisfying the conclusion of 3.8(ii). We say that $\mathcal{C}'$ is a *reflexive subcategory* of $\mathcal{C}$ if $\mathcal{C}'$ is a full subcategory and the inclusion $\mathcal{C}' \to \mathcal{C}$ has a left adjoint.

10/27/07

Examples 3.10.

(a) The subcategory **AbGp** is reflexive in **Gp**, as given an arbitrary group G we can let G' be the subgroup generated by all *commutators* $xyx^{-1}y^{-1}$. Then G/G' is abelian and any homomorphism $G \to A$ where A is abelian factors uniquely through $G \to G/G'$.

(b) The subcategory **tfAbGp** of torsion-free abelian groups is reflexive in **AbGp**: the reflector sends A to A/A_ϵ where A_ϵ is the torsion subgroup of A (i.e. the subgroup of all elements of finite order). Also, the subcategory **tAbGp** of torsion abelian groups is coreflexive in **AbGp**: the counit of this adjunction is the inclusion $A_\epsilon \hookrightarrow A$.

(c) The category **kHaus** of compact Hausdorff spaces is reflexive in **Top**: the reflector is the *Stone-Čech compactification* $X \mapsto \beta X$.

Lemma 3.11. *Suppose that we are given an equivalence of categories $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$ with F an isomorphism, $\alpha : 1_{\mathcal{C}} \to GF$, $\beta : FG \to 1_{\mathcal{D}}$. Then there exist natural isomorphisms $\alpha' : 1_{\mathcal{C}} \to GF$, $\beta' : FG \to 1_{\mathcal{D}}$ which satisfy the triangle identities so that $(F \dashv G)$ (and also $G \dashv F$).*

Proof. First note that

$$\begin{array}{ccc} 1_{\mathcal{C}} & \xrightarrow{\alpha} & GF \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle GF_\alpha} \\ GF & \xrightarrow{\alpha_{GF}} & GFGF \end{array}$$

commutes by naturality of α; but α is (pointwise) epic so $GF\alpha = \alpha_{GF}$. Similarly, $FG\beta = \beta_{FG}$. Now define $\alpha' = \alpha$ and let β' be the composite $FG \xrightarrow{\beta_{FG}^{-1}} FGFG \xrightarrow{(F\alpha_G)^{-1}} FG \xrightarrow{\beta} 1_{\mathcal{D}}$. To verify the triangle identities:

$$\begin{aligned} (G\beta')(\alpha'_G) &= G \xrightarrow{\alpha_G} GFG \xrightarrow{(G\beta_{FG})^{-1}} GFGFG \xrightarrow{(GF\alpha_G)^{-1}} GFG \xrightarrow{G\beta} G \\ &= G \xrightarrow{(G\beta)^{-1}} GFG \xrightarrow{\alpha_{GFG}} GFGFG \xrightarrow{\alpha_{GFG}^{-1}} GFG \xrightarrow{G\beta} G \\ &= G \xrightarrow{1_G} G \end{aligned}$$

where the second line follows by the naturality of α. Similarly,

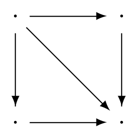
$$\begin{aligned} (\beta'_F)(F\alpha) &= F \xrightarrow{F\alpha} FGF \xrightarrow{\beta_{FGF}^{-1}} FGFGF \xrightarrow{(F\alpha_{GF})^{-1}} FGF \xrightarrow{\beta_F} F \\ &= F \xrightarrow{\beta_F^{-1}} FGF \xrightarrow{FGF\alpha} FGFGF \xrightarrow{(FGF\alpha)^{-1}} FGF \xrightarrow{\beta_F} F \\ &= F \xrightarrow{1_F} F \end{aligned}$$

by naturality of β. □

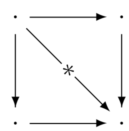
4. Limits

Definition 4.1. Let J be a category (almost always small or finite). By a *diagram of shape* J we mean a functor $D : J \to \mathcal{C}$. The objects $D(j)$ for $j \in \mathrm{ob}\, J$ are called *vertices* of D and the morphisms $D(\alpha)$ for $\alpha \in \mathrm{mor}\, J$ are called *edges* of D.

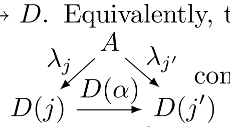
For example, if J is the finite category



a diagram of shape J is a commutative square. If J is the category



(where the starred arrow is meant to represent two parallel arrows) is a not-necessarily commutative square.

For any object A of $\mathcal{C}$ and any J we have a *constant diagram* ΔA of shape J all of whose vertices are A and all of whose edges are 1_A. By a cone over $D : J \to \mathcal{C}$ with *summit* A we mean a natural transformation $\lambda : \Delta A \to D$. Equivalently, this is a family $(\lambda_j : A \to D(j) \,|\, j \in \mathrm{ob}\, J)$ of morphisms (the *legs* of the cone) such that 

commutes for any $\alpha : j \to j'$ in J. Note that Δ is a functor $\mathcal{C} \to [J, \mathcal{C}]$ and a cone over D is an object of the arrow category $(\Delta \downarrow D)$. We say a cone $(\lambda_j : L \to D(j) \,|\, j \in \mathrm{ob}\, J)$ is a *limit* for D if it is a terminal object of $(\Delta \downarrow D)$.

Definition 4.2. We say that $\mathcal{C}$ *has limits* of shape J if $\Delta : \mathcal{C} \to [J, \mathcal{C}]$ has a right adjoint. By 3.3 this is equivalent to saying that every diagram $D : J \to \mathcal{C}$ has a limit.

Examples 4.3.

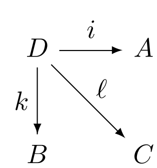
(a) If $J = \emptyset$ then $[J, \mathcal{C}]$ has a unique object and the category of cones over it is isomorphic to $\mathcal{C}$. So a limit for this diagram is a terminal object of $\mathcal{C}$ (and a colimit for it is an initial object).

(b) If J is a discrete category, a diagram of shape J is just a family of objects of $\mathcal{C}$, and a cone over it is a family of morphisms $(\lambda_j : A \to D(j) \,|\, j \in \mathrm{ob}\, J)$. So a limit for it is a *product* $\prod_{j \in \mathrm{ob}\, J} D(j)$. Similarly a colimit for this diagram is a *coproduct* $\sum_{j \in \mathrm{ob}\, J} D(j)$.

10/30/06

(c) Let J be the finite category $\cdot \rightrightarrows \cdot$ (so a diagram of shape J is a parallel pair $A \overset{f}{\underset{g}{\rightrightarrows}} B$). A cone over such a digram is of the form $A \xleftarrow{h} C \xrightarrow{k} B$ such that $fh = k = gh$, or equivalently a morphism $h : C \to A$ satisfying $fh = gh$. Thus a limit for the diagram is an equalizer for (f, g) (and a colimit for it is a coequalizer for (f, g)).

(d) Let J be the finite category $\cdot \longrightarrow \cdot \longleftarrow \cdot$. Then a diagram of shape J is a pair of morphisms $B \xrightarrow{g} C \xleftarrow{f} A$ with common codomain. A cone over this has the form



satisfying $fh = \ell = gk$ or equivalently a completion of the diagram to a commutative square. A terminal such completion is called a *pullback* for the pair (f, g). If $\mathcal{C}$ has products and equalizers then it has pullbacks: form the product $A \times B$ and then the equalizer $E \xrightarrow{e} A \times B \underset{g\pi_2}{\overset{f\pi_1}{\rightrightarrows}} C$. Then

$$\begin{array}{ccc} E & \xrightarrow{\pi_1 e} & A \\ {\scriptstyle \pi_2 e}\downarrow & & \\ B & & \end{array} \quad \text{is a limit for} \quad \begin{array}{ccc} & & A \\ & & \downarrow{\scriptstyle f} \\ B & \xrightarrow{g} & C \end{array}.$$

A colimit of shape J^{op} (i.e. of a diagram $C \xleftarrow{g} A \xrightarrow{f} B$) is called a *pushout* of (f, g).

Theorem 4.4.

(i) *If $\mathcal{C}$ has equalizers and all small (resp. all finite) products, then $\mathcal{C}$ has all small (resp. all finite) limits.*
(ii) *If $\mathcal{C}$ has pullbacks and a terminal object, then $\mathcal{C}$ has all finite limits.*

Proof.

(i) Let J be the small (resp. finite) and $D : J \to \mathcal{C}$ a diagram. Form the products $P = \prod_{j \in \mathrm{ob}\, J} D(j)$ and $Q = \prod_{\alpha \in \mathrm{mor}\, J} D(\mathrm{cod}\, \alpha)$. Now form $P \underset{g}{\overset{f}{\rightrightarrows}} Q$ defined by $\pi_\alpha f = \pi_{\mathrm{cod}\, \alpha}$ and $\pi_\alpha g = D(\alpha)\pi_{\mathrm{dom}\, \alpha}$, and the equalizer $e : E \to P$ of (f, g). We claim that $(\pi_j e : E \to D(j) \,|\, j \in \mathrm{ob}\, J)$ is a limit cone for J. It's a cone since for any edge $\alpha : j \to j'$ we have $D(\alpha)\pi_j e = \pi_\alpha g e = \pi_\alpha f e = \pi_j e$. If we are given any cone $(\lambda_i : A \to D(j) \,|\, j \in \mathrm{ob}\, J)$ we get a unique $\lambda : A \to P$ such that $\pi_j \lambda = \lambda_j$ for all j, but then $\pi_\alpha f \lambda = \pi_\alpha g \lambda$ for all α, so $f\lambda = g\lambda$. So λ factors uniquely as μe, so μ is the unique factorization of $(\lambda_j \,|\, j \in \mathrm{ob}\, J)$ through $(\pi_j e \,|\, j \in \mathrm{ob}\, J)$.
(ii) It suffices to construct finite products and equalizers in $\mathcal{C}$. We can construct the product $A \times B$ as the pullback of $B \longrightarrow * \longleftarrow *A$ where $*$ is the terminal object, and then construct $\prod_{i=1}^n A_i$ as $(\cdots((A_1 \times A_2) \times A_3) \cdots \times A_{n-1}) \times A_n$. We can form the equalizer of $f, g : A \to B$ as the pullback of $A \xrightarrow{(f,g)} B \times B \xleftarrow{(1_B, 1_B)} B$, since a cone over this diagram consists of $A \xleftarrow{h} C \xrightarrow{k} B$ satisfying $fh = 1_B k$ and $gh = 1_B k$.

□

Definition 4.5. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor, J a (small) category.

- We say F *preserves limits* of shape J if, given $D : J \to \mathcal{C}$ and a limit cone $(\lambda_j : L \to D(j) \,|\, j \in \mathrm{ob}\, J)$ the cone $(F\lambda_j : FL \to FD(j) \,|\, j \in \mathrm{ob}\, J)$ is a limit cone for FD in $\mathcal{D}$.
- We say F *reflects limits* of shape J if given $D : J \to \mathcal{C}$ and a cone $(\lambda_j : L \to D(j) \,|\, j \in \mathrm{ob}\, J)$ such that $(F\lambda_j : FL \to FD(j) \,|\, j \in \mathrm{ob}\, \mathcal{C})$ is a limit for FD, then the original cone was a limit for D.
- We say that F *creates limits* of shape J if, given $D : J \to \mathcal{C}$ and a limit $(\mu_j : M \to FD(j) \,|\, j \in \mathrm{ob}\, J)$ for FD, there exists a cone $(\lambda_j : L \to D(j) \,|\, j \in \mathrm{ob}\, J)$ over D mapping to a limit for FD, and any such cone is a limit in $\mathcal{C}$. (Note that if we require M to be in the image of F then category equivalences might not create limits, as M may not be in the image of the equivalence. This definition says that if there is a limit for FD in $\mathcal{D}$ then there is a limit for D in $\mathcal{C}$ that maps to a limit of FD in $\mathcal{D}$.)

Corollary 4.6. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. In any version of the above theorem 4.4 we may replace "$\mathcal{C}$ has" by either "$\mathcal{C}$ has and F preserves" or "$\mathcal{D}$ has and F creates."*

Examples 4.7.

(a) $U : \mathbf{Gp} \to \mathbf{Set}$ creates all small limits, but doesn't preserve or create colimits.
(b) $U : \mathbf{Top} \to \mathbf{Set}$ preserves all limits and colimits, but doesn't reflect them.
(c) $U : \mathcal{C}/B \to \mathcal{C}$ creates colimits, since a digram $D : J \to \mathcal{C}/B$ is the same thing as a diagram $UD : J \to \mathcal{C}$ together with a cone $(UD(g) \to B \,|\, j \in \mathrm{ob}\, J)$. So, given a colimit $(\lambda_j : UD(j) \to L \,|\, j \in \mathrm{ob}\, J)$ in $\mathcal{C}$ we get a unique $h : L \to B$; if the λ_j are all morphisms $D(j) \to h$ in $\mathcal{C}/B$, they form a cone under D and it's a colimit cone. But $U : \mathcal{C}/B \to \mathcal{C}$ doesn't preserve or reflect products: the product

of $f : A \to B$ and $g : C \to B$ in $\mathcal{C}/B$ is the diagonal of the pullback square $\begin{array}{ccc} P & \longrightarrow & A \\ \downarrow & & \downarrow f \\ C & \xrightarrow{g} & B \end{array}$ in $\mathcal{C}$, which is not necessarily a product of A and B in $\mathcal{C}$, (consider, for example, **Set** with $B \neq \{1\}$).

11/1/06

(d) Let $\mathcal{C}$ and $\mathcal{D}$ be categories. The forgetful functor $[\mathcal{C}, \mathcal{D}] \to \mathcal{D}^{\text{op}}$ creates all limits and colimits which exist in $\mathcal{D}$.

To prove this, let $D : J \to [\mathcal{C}, \mathcal{D}]$ be a diagram; we consider it as a functor $J \times \mathcal{C} \to \mathcal{D}$. For each $A \in \text{ob}\,\mathcal{C}$ we can form a limit cone $(\lambda_{j,A} : LA \to D(j, A) \mid j \in \text{ob}\, J)$ for $D(-, A) : J \to \mathcal{D}$. For each $f : A \to B$ in $\mathcal{C}$ the composites

$$LA \xrightarrow{\lambda_{j,A}} D(j, A) \xrightarrow{D(j,f)} D(j, B) \qquad j \in \text{ob}\, J$$

form a cone over $D(-, B)$ and induce a unique $Lf : LA \to LB$ such that $\lambda_{j,B} Lf = D(j, f)\lambda_{j,A}$ for all j.

Given $g : B \to C$, $L(gf)$ and $(Lg)(Lf)$ are factorizations of the same cone through a limit so they are equal; hence L is a functor $C \to \mathcal{D}$ and each $\lambda_{j,-}$ is a natural transformation $L \to D(j, -)$. The $(\lambda_{j,-} \mid j \in \text{ob}\, J)$ also form a cone over D (regarded as a diagram of shape J in $[\mathcal{C}, \mathcal{D}]$) with summit L.

In order to finish this proof we need to check that this is a limit cone. To do this we take any other cone over D and consider its image for a given element $A \in \mathcal{C}$ and construct the natural transformation to the above limit.

(e) The inclusion functor $\mathbf{AbGp} \to \mathbf{Gp}$ reflects coproducts but doesn't preserve them. A free product (which is a free product in **Gp**) $G * H$ is never abelian unless one of G and H is the trivial group, but in that event it is also a coproduct in **AbGp**.

Remark. A morphism $f : A \to B$ in any category is a monomorphism iff

$$\begin{array}{ccc} A & \xrightarrow{1_A} & A \\ {\scriptstyle 1_A}\downarrow & & \downarrow{\scriptstyle f} \\ A & \xrightarrow[f]{} & B \end{array}$$

is a pullback. Hence a functor which preserves/reflects pullbacks will also preserve/reflect monomorphisms. To see this, note that if the above diagram is a pullback then any cone $A \xleftarrow{k} C \xrightarrow{h} A$ satisfyling $fh = fk$ must satisfy $h = k$. Conversely if f is a monomorphism then any cone over $A \xrightarrow{f} B \xleftarrow{f} A$ has both legs equal and so factors (necessarily uniquely) through $A \xleftarrow{1_A} A \xrightarrow{1_A} A$.

Theorem 4.8. *Suppose $G : \mathcal{D} \to \mathcal{C}$ has a left adjoint $F : \mathcal{C} \to \mathcal{D}$. Then G preserves all limits which exist in $\mathcal{D}$.*

Proof 1. Suppose that $\mathcal{C}$ and $\mathcal{D}$ both have limits of some shape J. Then the diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ {\scriptstyle \Delta}\downarrow & & \downarrow{\scriptstyle \Delta} \\ [J, \mathcal{C}] & \xrightarrow{[J,F]} & [J, \mathcal{D}] \end{array}$$

commutes and all the functors in it have right adjoints. So by corollary 3.6

$$\begin{array}{ccc} \mathcal{C} & \xleftarrow{G} & \mathcal{D} \\ {\scriptstyle \lim_J}\uparrow & & \uparrow{\scriptstyle \lim_J} \\ [J, \mathcal{C}] & \xleftarrow{[J,G]} & [J, \mathcal{D}] \end{array}$$

commutes up to natural isomorphism. But this means exactly that G preserves limits of shape J. □

Proof 2. Let $D : J \to \mathcal{D}$ and let $(\lambda_j : L \to D(j) \,|\, j \in \text{ob}\, J)$ be a limit for it. Given a cone $(\mu_j : A \to GD(j) \,|\, j \in \text{ob}\, J)$ over GD in $\mathcal{C}$ we get a family of morphisms $(\overline{\mu}_j : FA \to D(j) \,|\, j \in \text{ob}\, J)$ which form a cone over D by naturality of $\mu \mapsto \overline{\mu}$. So we get a unique $\overline{\mu} : FA \to L$ such that $\lambda_j \overline{\mu} = \overline{\mu}_j$ for each j, i.e. a unique $\mu : A \to GL$ such that $(G\lambda_j)\mu = \mu_j$. Thus the $G\lambda_j$ are a limit cone. □

Our aim now is to show that if $\mathcal{D}$ has and $G : \mathcal{D} \to \mathcal{C}$ preserves "all" limits then G has a left adjoint.

Lemma 4.9. *Suppose $\mathcal{D}$ has and $G : \mathcal{D} \to \mathcal{C}$ preserves limits of shape J. Then $(A \downarrow G)$ has limits of shape J for each $A \in \text{ob}\, \mathcal{C}$ and $U : (A \downarrow G) \to \mathcal{D}$ creates them.*

Proof. Suppose that we are given $D : J \to (A \downarrow G)$. We can consider D as a cone $(f_i : A \to GUD(j))$ over $GUD : J \to \mathcal{C}$. So if $(\lambda_j : L \to UD(j) \,|\, j \in \text{ob}\, J)$ is a limit for UD then we get a unique $f : A \to GL$ such that $(G\lambda_j)f = f_j$ for each j, i.e. such that each λ_j is a morphism $(L, f) \to (UD(j), f_j)$ in $(A \downarrow G)$.

The λ_j form a cone over D with summit (L, f), since they form a cone over UD and U is faithful. Given any cone $(\mu_j : (B, g) \to (UD(j), f_j))$ over D in $(A \downarrow G)$ the μ_j also form a cone over UD with summit B so they induce a unique $\mu : B \to L$ such that $\lambda_j \mu = \mu_j$ for al j. We need to show that $(G\mu)g = f$, but these are factorizations of the same cone over GUD through GL so they are equal. So $\mu : (B, g) \to (L, f)$ in $(A \downarrow G)$ and it is the unique factorization of $(U_j, 1_j)$ through $(\lambda_j, 1_j)$ in this category. Thus $(A \downarrow G)$ has limits of shape J. □

11/3/06

Lemma 4.10. *Specifying an initial object for a category $\mathcal{C}$ is equivalent to specifying a limit for $1_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$.*

Proof. If I is an initial object the unique morphisms $(I \to A \,|\, A \in \text{ob}\, \mathcal{C})$ form a cone over $1_{\mathcal{C}}$. Given any cone $(\lambda_A : S \to A \,|\, A \in \text{ob}\, \mathcal{C})$ over $1_{\mathcal{C}}$ $\lambda_I : S \to I$ is a factorization through the one with summit I, so the cone with summit I is a limit cone over $1_{\mathcal{C}}$.

Suppose that we are given a limit cone $(\lambda_A : L \to A \,|\, A \in \text{ob}\, \mathcal{C})$ for $1_{\mathcal{C}}$. We need to show that, for each A, λ_A is the unique morphism $L \to A$. Given $f : L \to A$ we have $f\lambda_L = \lambda_A$. In particular, $\lambda_A \lambda_L = \lambda_A$ for all A, so λ_L is a factorization of the limit cone through itself. So $\lambda_L = 1_L$ and λ_A is the unique map $L \to A$. □

Theorem 4.11 (Primitive adjoint functor theorem)**.** *If $\mathcal{D}$ has and $G : \mathcal{D} \to \mathcal{C}$ preserves all limits then G has a left adjoint.*

Proof. By lemma 4.9, each $(A \downarrow G)$ has all limits. Therefore, by lemma 4.10, each $(A \downarrow G)$ has an initial object. By theorem 3.3 we then see that G has a left adjoint. □

We call a category $\mathcal{C}$ *complete* if it has all small limits.

Theorem 4.12 (General adjoint functor theorem)**.** *Let $\mathcal{D}$ be locally small and complete, and let $G : \mathcal{D} \to \mathcal{C}$ be a functor. Then G has a left adjoint iff G preserves all small limits and satisfies the "solution set condition": for every $A \in \text{ob}\, \mathcal{C}$ there exists a set of morphisms $\{f_i : A \to GB_i \,|\, i \in I\}$ such that every $f : A \to GB$ factors as $A \xrightarrow{f_i} GB_i \xrightarrow{Gh} GB$ for some $i \in I$ and some $h : B_i \to B$ in $\mathcal{D}$.*

The set $\{f_i : A \to GB_i \,|\, i \in I\}$ is called the *solution set*.

Proof. For the forward direction, note that G preserves limits by theorem 4.8, and $\{q_A : A \to GFA\}$ is a solution set for A by theorem 3.3.

For the backwards direction, note that each $(A \downarrow G)$ is complete by lemma 4.9 and it inherits local smallness from $\mathcal{D}$. So it suffices to show that if a category $\mathcal{A}$ is locally small, complete and has a solution set of objects then it has an initial object. Let $\{C_i \,|\, i \in I\}$ be a solution set of objects. Form $P = \prod_{i \in I} C_i$ and let $e : E \to P$ be the limit of the diagram with one object (P) and whose edges are all of the morphisms $P \to P$ in $\mathcal{A}$. For every object D we have a morphism $P \to C_i \to D$ for some $i \in I$, and hence a morphism $E \to P \to D$. Suppose we have $f, g : E \to D$. Form their equalizer $h : F \to E$. There exists some $k : P \to F$ and the composite ehk is an endomoprhism of P. So by definition of E $ehke = 1_P e$, and as e is a monomorphism $hke = 1_E$. In particular h is epic, so $f = g$. Thus E is an initial object of $\mathcal{A}$ and we are done. □

Lemma 4.13. *Suppose that we are given a pullback square* $\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ C & \xrightarrow[k]{} & D \end{array}$ *with h monic. Then g is monic.*

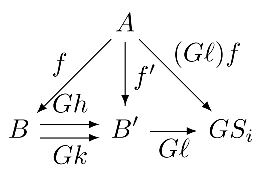
Proof. Suppose that $\ell, m : E \to A$ satisfies $g\ell = gm$. Then $hf\ell = kg\ell = kgm = hfm$. As h is monic we see that $f\ell = fm$. So ℓ and m are factorizations of the same cone through a limit, hence $\ell = m$. □

Definition 4.14. A *subobject* of A in a category is a monomorphism $A' \hookrightarrow A$. We say a category $\mathcal{C}$ is *well-powered* if for every $A \in \operatorname{ob} \mathcal{C}$ there exists a set of suboebjects $\{A_i \hookrightarrow A \mid i \in I\}$ such that every $A' \hookrightarrow A$ is isomorphic (in $\mathcal{C}/A$) to some $A_i \hookrightarrow A$.

For example, **Set**, **Gp**, **Top** are all well powered.

Theorem 4.15 (Special adjoint functor theorem)**.** *Suppose that $\mathcal{C}$ is locally small and that $\mathcal{D}$ is locally small, complete, and well-powered, and that $\mathcal{D}$ has a coseparating set of objects. Then a functor $G : \mathcal{D} \to \mathcal{C}$ has a left adjoint iff G preserves all small limits.*

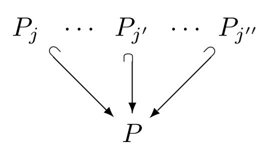
Proof. The forward direction follows from 4.12. For the backward direction we first show that each $(A \downarrow G)$ is complete, locally small and well powered and has a coseparating set. Completeness and local smallness are proven as before. For well-poweredness, note that a morphism $h : (B', f') \to (B, f)$ in $(A \downarrow G)$ is monic iff it is monic in $\mathcal{D}$, so subobjects of (B, f) in $(A \downarrow G)$ correspond to subobjects $m : B' \hookrightarrow B$ such that f factors (uniquely) through $Gm : GB' \hookrightarrow GB$. So, up to isomorphism, these form a set. For the coseparating set, let $\{S_i \mid i \in I\}$ be a coseparating set for $\mathcal{D}$. Then the set $\{(S_i, f) \mid i \in I,\ f \in \mathcal{C}(A, GS_i)\}$ is a coseparating set for $(A \downarrow G)$, since if we have



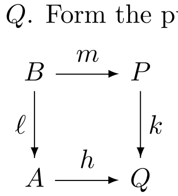
with $h \neq k$ there exists some $\ell : B' \to S_i$ with $\ell h \neq \ell k$ and ℓ is a morphism $(B', f') \to (S_i, (G\ell)f')$ in $(A \downarrow G)$.

11/6/06

It remains to show that if $\mathcal{A}$ is complete, locally small and well powered and has a corresponding set $\{S_i \mid i \in I\}$ of objects then it has an initial object. Form $P = \prod_{i \in I} S_i$. Let $\{P_j \to P \mid j \in J\}$ be a representative set of subobjects of P and form the limit of the diagram



whose objects are all the $P_j \hookrightarrow P$ for $j \in J$. If L is the summit of the limit cone then $L \to P$ is monic (by the same argument as above) and it is the smallest subobject of P since it factors through every $P_j \hookrightarrow P$. We claim that L is an initial object of $\mathcal{A}$. Suppose we had two maps $f, g : L \to A$. Then we could form their equalizer $E \hookrightarrow L$, but $E \hookrightarrow L \hookrightarrow P$ is monic, so $L \hookrightarrow P$ factors through it and hence J_L factor through $E \hookrightarrow L$, so $E \hookrightarrow L$ is epic and $f = g$. Thus we have at most one map $L \to A$ for each A. In order to show existence, suppose that we are given $A \in \operatorname{ob} \mathcal{A}$. Consider $K = \{(i, f) \mid i \in I,\ f : A \to S_i\}$ and form $Q = \prod_{(i,f) \in K} S_i$. We have a canonical $h : A \to Q$ defined by $h = \prod_{(i,f)} f$, and h is monic. Since the S_i form a separating family we similarly have $k : P \to Q$. Form the pullback



Then m is monic, so $L \hookrightarrow P$ factors through it and we have a morphism $L \to B \to A$. □

Examples 4.16.

(a) If we didn't know how to construct free groups we could use GAFT to construct a left adjoint for $U : \mathbf{Gp} \to \mathbf{Set}$. We already know that $\mathbf{Gp}$ has an U preserves all small limits. So we need only to verify the solution set conclusion. Given a set A any function $\lambda : A \to UG$ factors as $A \to UG' \to UG$ where G' is the subgroup generated by $\{f(a) \mid a \in A\}$. We take a set of $|G'|$ and equip all subsets of it with all possible group structures, plus all possible maps from A to obtain a solution set.

(b) Consider the category $\mathbf{cLat}$ of complete lattices and the forgetful functor $U : \mathbf{cLat} \to \mathbf{Set}$. Just as for groups, $\mathbf{cLat}$ has and U preserves all small limits, and $\mathbf{cLat}$ is locally small. However, AW Hales showed that there does not exist a free complete lattice on three generators, so the solution set condition fails for $A = \{1, 2, 3\}$ and U doesn't have a left adjoint.

(c) Consider the inclusion functor $I : \mathbf{kHaus} \to \mathbf{Top}$. $\mathbf{kHaus}$ has small products and A preserves them. It has equalizers because, given $f, g : X \to Y$ with Y Hausdorff, their equalizer is a closed subspace of X and hence compact if X is. $\mathbf{kHaus}$ and $\mathbf{Top}$ are locally small. $\mathbf{kHaus}$ is well-powered since subobjects of X are all isomorphic to closed subspaces of X. By Uruson's lemma the closed interval $[0, 1]$ is a coseparator for $\mathbf{kHaus}$. So by 4.15 I has a left adjoint β. The Stone-Čech compactification functor. Čech's original (1937) construction of βX was as follows: form $P = \prod_{f:X\to[0,1]} [0, 1]$ and then form the closure of the image of the canonical map $X \to P$. (Note: this is precisely what the SAFT tells you to do.)

5. Monads

Suppose that we are given an adjunction $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$, with $(F \dashv G)$. What properties does the "trace" of the adjunction have as a functor on the category $\mathcal{C}$? We have the functor $T = GF : \mathcal{C} \to \mathcal{C}$ and the unit $\eta : 1_{\mathcal{C}} \to T$. We also have a natural transformation $\mu = G\epsilon_F : \Pi = GFGF \to GF$. From the triangular identities for η and ϵ we get the identities

$$\begin{array}{ccc} T & \xrightarrow{T\eta} & TT \\ & \searrow^{1_T} & \downarrow \mu \\ & & T \end{array} \qquad \begin{array}{ccc} T & \xrightarrow{\eta_T} & TT \\ & \searrow^{1_T} & \downarrow \mu \\ & & T \end{array}$$

and from the naturality of ϵ we get the commutativity of

$$\begin{array}{ccc} TTT & \xrightarrow{T\mu} & TT \\ \mu_T \downarrow & & \downarrow \mu \\ TT & \xrightarrow[\mu]{} & T \end{array}$$

Definition 5.1. By a *monad* $\Pi = (T, \eta, \mu)$ on a category $\mathcal{C}$ we mean a functor $T : \mathcal{C} \to \mathcal{C}$ equipped with a natural transformation $\eta : 1_{\mathcal{C}} \to T$ and $\mu : TT \to T$ satisfying the above three diagrams. Any adjuncation $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ induces a monad $(GF, \eta, G\epsilon_F)$ on $\mathcal{C}$ and a comonad $(G, \epsilon, F\eta_G)$ on $\mathcal{D}$

11/8/06

Example 5.2. Given a monoid M, the functor $M \otimes - : \mathbf{Set} \to \mathbf{Set}$ has a monad structure with unit $\eta_A : A \to M \times A$ sending a to (e, a) and multiplication $\mu_A : M \times M \times A \to M \times A$ sending (m, n, a) to (mn, a). This monad is induced by an adjunction $F : \mathbf{Set} \rightleftarrows M \times \mathbf{Set}$ where $M \times \mathbf{Set}$ is the category of sets with an M-action, G is the forgetful functor and $FA = M \times A$ (with M action by multiplication on the left factor).

Definition 5.3. Let $\Pi = (T, \eta, \mu)$ be a monad on a category $\mathcal{C}$. By a Π-algebra we mean a pair (A, α) where $A \in \operatorname{ob} \mathcal{C}$ and $\alpha : TA \to A$ satisfying

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & TA \\ & \searrow^{1_A} & \downarrow \alpha \\ & & A \end{array} \quad \text{and} \quad \begin{array}{ccc} TTA & \xrightarrow{T\alpha} & TA \\ \mu_A \downarrow & & \downarrow \alpha \\ TA & \xrightarrow{\alpha} & A \end{array}.$$

A *homomorphism* of Π-algebras is a morphism $f : A \to B$ such that

$$\begin{array}{ccc} TA & \xrightarrow{Tf} & TB \\ \alpha\downarrow & & \downarrow\beta \\ A & \xrightarrow[f]{} & B \end{array}$$

commutes. We write $\mathcal{C}^\Pi$ for the category of Π-algebras and homomorphisms between them (and call it the *Eilenberg-Moore* category of Π). There's an obvious forgetful functor $G^\Pi : \mathcal{C}^\Pi \to \mathcal{C}$ sending (A, α) to A and f to f.

Lemma 5.4. *G^Π has a left adjoint F^Π and the monad induced by $(F^\Pi \dashv G^\Pi)$ is Π.*

Proof. We define $F^\Pi A = (TA, \mu_A)$. This is a Π-algebra by two of the commutative diagrams in the definition of Π. And we define $F^\Pi(A \to B) = Tf : (TA, \mu_A) \to (TB, \mu_B)$, which is a homomorphism by the naturality of μ. To verify that $(F^\Pi \dashv G^\Pi)$ we construct the unit and counit of the adjunction. $G^\Pi F^\Pi = T$ so we take $\eta : 1 \to T$ as the unit. We define $\epsilon_{(A,\alpha)} = \alpha$: the associativity condition for α says that this is a homomorphism $F^\Pi G^\Pi(A, \alpha) \to (A, \alpha)$ and naturality follows from the definition of homomorphism. The identity$(G^\Pi\alpha)\eta_A = 1_{G^\Pi(A,\alpha)}$ is the unit condition on α. The identity $(\epsilon_{FA})(F\eta_A) = 1_{FA}$ is the condition that $\mu_A\eta_A = 1_{TA}$ which is included in the definition of a monad. □

Note that if $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ with $(F \dashv G)$ is an adjunction inducing Π we could replace $\mathcal{D}$ by the full subcategory of $\mathcal{D}$ of objects of the form FA. So in trying to construct $\mathcal{D}$ we may assume F is surjective on objects. Also, morphisms $FA \to FB$ in $\mathcal{D}$ correspond to morphisms $A \to GFB = TB$ in $\mathcal{C}$.

Definition 5.5. Let $\Pi = (T, \eta, \mu)$ be a monad on $\mathcal{C}$. The *Kleisli category* $\mathcal{C}_\Pi$ is defined by $\mathrm{ob}\,\mathcal{C}_\Pi = \mathrm{ob}\,\mathcal{C}$. Morphisms $A \to B$ in $\mathcal{C}_\Pi$ are morphisms $A \to TB$ in $\mathcal{C}$. The composite of $A \xrightarrow{f} B \xrightarrow{g} C$ is $A \xrightarrow{f} TB \xrightarrow{Tg} TTC \xrightarrow{\mu_C} TC$ and the identity morphism $A \to A$ is η_A.

To verify associativity suppose we are given $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$. Then

$$\begin{aligned} h(gf) &= A \xrightarrow{f} TB \xrightarrow{Tg} TTC \xrightarrow{\mu_C} TC \xrightarrow{Th} TTD \xrightarrow{\mu_D} TD \\ &= A \xrightarrow{f} TB \xrightarrow{Tg} TTC \xrightarrow{TTh} TTTD \xrightarrow{\mu_{TD}} TTD \xrightarrow{\mu_D} TD \\ &= A \xrightarrow{f} TB \xrightarrow{Tg} TTC \xrightarrow{TTh} TTTD \xrightarrow{T\mu_D} TTD \xrightarrow{\mu_D} TD \\ &= A \xrightarrow{f} TB \xrightarrow{T(hg)} TTD \xrightarrow{\mu_D} TD \\ &= (hg)f \end{aligned}$$

where the second line follows by naturality of μ and the third by the associativity of μ. For the unit law

$$\begin{aligned} f\eta_A &= A \xrightarrow{\eta_A} TA \xrightarrow{Tf} TTB \xrightarrow{\mu_B} TB \\ &= A \xrightarrow{f} TB \xrightarrow{\eta_{TB}} TTB \xrightarrow{\mu_B} TB \\ &= A \xrightarrow{f} TB \end{aligned}$$

by one of the unit laws for Π. The other unit law is analogous.

Lemma 5.6. *There is an adjunction $F_\Pi : \mathcal{C} \rightleftarrows \mathcal{C}_\Pi : G_\Pi$ inducing the monad Π.*

Proof. We define $F_\Pi A = A$ and $F_\Pi(f) = \eta_B f$ for $f : A \to B$, and we define $G_\Pi(A) = TA$ and $G_\Pi(f) = (\mu_B)(Tf)$ for $f : A \to B$. We will construct the unit and counit of this adjunction to see that this does, in fact, induce Π.

To verify that F_Π is a functor suppose that we are given $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathcal{C}$. Then

$$\begin{aligned}(F_\Pi g)(F_\Pi f) &= A \xrightarrow{f} TB \xrightarrow{Tg} TC \xrightarrow{T\eta_C} TTC \xrightarrow{\mu_C} TC \\ &= A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{\eta_C} TC \\ &= F_\Pi(gf)\end{aligned}$$

To verify that G_Π is a functor note that $G_\Pi(\eta_A) = \mu_A T\eta_A = 1_{TA}$. For $f : A \to B$ and $g : B \to C$

$$\begin{aligned}G_\Pi(gf) &= TA \xrightarrow{Tf} TTB \xrightarrow{TTg} TTTC \xrightarrow{T\mu_C} TTC \xrightarrow{\mu_C} TC \\ &= TA \xrightarrow{Tf} TTB \xrightarrow{TTg} TTTC \xrightarrow{\mu_{TC}} TTC \xrightarrow{\mu_C} TC \\ &= TA \xrightarrow{Tf} TTB \xrightarrow{\mu_B} TB \xrightarrow{Tg} TTC \xrightarrow{\mu_C} TC \\ &= (G_\Pi g)(G_\Pi f).\end{aligned}$$

As $G_\Pi F_\Pi(f) = Tf$ we can take the unit of the adjunction to be η. Since $F_\Pi G_\Pi A = TA$ we take the counit ϵ_A to be 1_{TA}.

11/10/06

We need to check that the counit is natural; in particular, we need to check that

$$\begin{array}{ccc} F_\Pi G_\Pi A & \xrightarrow{F_\Pi G_\Pi f} & F_\Pi G_\Pi B \\ {\scriptstyle 1_{TA}}\downarrow & & \downarrow{\scriptstyle 1_{TB}} \\ A & \xrightarrow{f} & B \end{array}$$

commutes. As

$$F_\Pi G_\Pi(f) = TA \xrightarrow{Tf} TTB \xrightarrow{\mu_B} TB \xrightarrow{\eta_B} TTB$$

the top composite is $TA \xrightarrow{Tf} TTB \xrightarrow{\mu_B} TB\eta_{TB}TTB \xrightarrow{1_{TTB}} TTB \xrightarrow{\mu_B} TB$; note that the composition of the last three functions is 1_{TB} so this is simply $\mu_B(Tf)$, which is the bottom composite by definition.

It remains to show that η and ϵ satisfy $\epsilon_{F_\Pi}(F_\Pi\eta) = 1_{F_\Pi}$ and $(G_\Pi\epsilon)\eta_{G_\Pi} = 1_{G_\Pi}$. The first of these is

$$A \xrightarrow{\eta_A} TA \xrightarrow{\eta_{TA}} TTA \xrightarrow{1_{TTA}} TTA \xrightarrow{\mu_A} TA$$

which is simply η_A, exactly the image of the identity under F_Π. The second of these is just one of the triangle conditions on η and μ. □

Given a monad Π on $\mathcal{C}$, let $\mathbf{Adj}(\Pi)$ denote the category whose objects are adjuctions $\mathcal{C} \rightleftarrows \mathcal{D}$ inducing Π and whose morphisms $(F \dashv G) \to (F' \dashv G')$ are functors $k : \mathcal{D} \to \mathcal{D}'$ such that $kF = F'$ and $G'k = G$.

Theorem 5.7. *The Kleisli adjunction* $(F_\Pi \dashv G_\Pi)$ *is initial in* $\mathbf{Adj}(\Pi)$ *and the Eilenberg-Moore adjunction* $(F^\Pi \dashv G^\Pi)$ *is terminal.*

Proof. Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ be an arbitrary object of $\mathbf{Adj}(\Pi)$; let ϵ be the counit of the adjunction. We define $k : \mathcal{D} \to \mathcal{C}^\Pi$ by $kB = (GB, G\epsilon_B)$: note that $(G\epsilon_B)\eta_{GB} = 1_{GB}$ and $(G\epsilon_B)(G\epsilon_{FGB}) = (G\epsilon_B)(GFG\epsilon_B)$ by naturality of ϵ. And $k(g : B \to B') = Gg$ (which is an algebra homomorphism since ϵ is natural). clearly $G^\Pi k = G$ and $kFA = (GFA, G\epsilon_{FA}) = (TA, \mu_A) = F^\Pi A$, and $kF(f : A \to A') = GFf = Tf = F^\Pi f$. If $k : \mathcal{D} \to \mathcal{C}^\Pi$ satisfies $G^\Pi k' = G$ and $k'F = F^\Pi$ then necessarily $k'B = (GB, \beta_B)$ and $k'g = Gg$ for some $\beta : FGB \to B$ in $\mathcal{D}$ yielding $\beta_{FGB} = \mu_{GB} = G\epsilon_{FGB}$ and $\beta_B(GFG\epsilon_B) = (G\epsilon_B)(G\epsilon_{FGB})$. But this would still hold with β_B replaced by $G\epsilon_B$ and $GFG\epsilon_B$ is split epic (a.k.a has a right inverse) so $\beta_B = G\epsilon_B$.

Now define $L : \mathcal{C}_\Pi \to \mathcal{D}$ by $LA = FA$ and $Lf = \epsilon_{FA'}Ff$ for $f : A \to A'$. To check that L is a functor

$$LF_\Pi(f : A \to A') = FA \xrightarrow{Ff} FA' \xrightarrow{F\eta_{A'}} FGFA' \xrightarrow{\epsilon_{FA'}} = Ff.$$

$GLA = GFA = TA = G_\Pi A$. $GL(f) = (G\epsilon_{FA'})(GFf) = Tf\mu_{A'} = G_\Pi f$. We need to also check uniqueness. □

Theorem 5.8. *Let* Π *be a monad on* $\mathcal{C}$. *Then*

(i) $G^\Pi : \mathcal{C}^\Pi \to \mathcal{C}$ *creates limits of all shapes which exist in* $\mathcal{C}$

(ii) G^Π *creates colimits of shape* J *iff* T *preserves them.*

Proof.

(i) Suppose we are given $D : J \to \mathcal{C}^\Pi$ and suppose $G^\Pi D$ has a limit $(\lambda_j : L \to G^\Pi D(j) \,|\, j \in \operatorname{ob} J)$ in $\mathcal{C}$. Write $D(j)$ as $(GD(j), \delta_j)$. Then the $T\lambda_j$ form a cone over TGD and the δ_j form a natural transformation $TGD \to GD$, so the composites $(\delta_j)(T\lambda_j)$ form a cone over GD. Hence we get a unique $\theta : TL \to L$ such that $\lambda_j\theta = \delta_j(T\lambda_j)$ for each j. We claim that (L, θ) is a Π-algebra. To verify (e.g.) the associativity axiom we have to show euqality of two morphisms $TTL \rightrightarrows L_j$ but their composites with each λ_j can be factored as $TTL \xrightarrow{TT\lambda_j} TTGD(j) \underset{g}{\overset{f}{\rightrightarrows}} GD(j)$ where $f = g$ since $D(j)$ is an algebra. If we're given any cone $(\mu_j : M \to D(j) \,|\, j \in \operatorname{ob} J)$ in $\mathcal{C}^\Pi$ we get a unique factorization $\mu_j = \lambda_j\varphi$ for a unique $\varphi : GM \to L$ in $\mathcal{C}$ and φ is an algebra homomorphism $M \to (L, \theta)$ by the same argument as before.

(ii) To see the forward direction, note that if G^Π creates colimits of shape J then $T = G^\Pi F^\Pi$ preserves them since F^Π preserves all colimits that exist. For the backwards direction copy the argument of (i) but use the fact that if L is the summit of a colimit cone then so are TL and TTL.

□

11/13/06

Definition 5.9. We say an adjunction $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ (with induced monad Π) is *monadic* if the comparison functor $K : \mathcal{D} \to \mathcal{C}^\Pi$ is part of an equivalence. We also say $G : \mathcal{D} \to \mathcal{C}$ is *monadic* if it has a left adjoint and the adjunction is monadic.

Given an adjunction $(F \dashv G)$, for any object B of $\mathcal{D}$ we have a diagram

$$FGFGB \underset{\epsilon_{FGB}}{\overset{FG\epsilon_B}{\rightrightarrows}} FGB \xrightarrow{\epsilon_B} B$$

(called the *standard free presentation of* B); the monacity theorems all use the idea that $\mathcal{C}^\Pi$ is characterized in $\mathbf{Adj}(\Pi)$ by the fact that this diagram is a coequalizer for any B.

Definition 5.10.

(i) We say a parallel pair $f, g : A \to B$ is *reflexive* if there exists $r : B \to A$ such that $fr = gr = 1_B$. By a *reflexive coequalizer* we mean a coequalizer of a reflexive pair.

(ii) We say a diagram

$$A \underset{\overset{\longleftarrow}{t}}{\overset{f}{\underset{g}{\rightrightarrows}}} B \underset{s}{\overset{h}{\rightleftarrows}} C$$

is a *split coequalizer diagram* if it satisfies $hf = hg$, $hs = 1_C$, $gt = 1_B$ and $ft = sh$. If these hold then h is indeed a coequalizer of f and g: if $k : B \to D$ satisfies $kf = kg$ then $k = kgt = kft = ksh$ so k factors through h and this factorization is unique since h is a split epic.

(iii) Given $G : \mathcal{D} \to \mathcal{C}$ we say a parallel pair $f, g : A \to B$ is *G-split* if Gf, Gg are part of a split coequalizer diagram in $\mathcal{C}$. Note that the standard free presentation $FG\epsilon_B, \epsilon_{FGB} : FGFGB \to FGB$ is reflexive with common splitting $F\eta_{GB}$, and also G-split since

$$GFGFGB \underset{\overset{\longleftarrow}{\eta_{GFGB}}}{\overset{GFG\epsilon_B}{\underset{G\epsilon_{FGB}}{\rightrightarrows}}} GFGB \underset{GB}{\overset{G\epsilon_B}{\rightleftarrows}} GB$$

is a split coequalizer diagram.

Theorem 5.11 (Precise Monadicity Theorem)**.** *Let $G : \mathcal{D} \to \mathcal{C}$ be a functor. Then G is monadic iff*

(i) *G has a left adjoint*

(ii) *G creates coequalizers of G-split pairs.*

Theorem 5.12 (Crude Monadicity Theorem)**.** *Let $G : \mathcal{D} \to \mathcal{C}$ be a functor and suppose*

(i) *G has a left adjoint,*
(ii) *G reflects isomorphisms*
(iii) *$\mathcal{D}$ has and G preserves coequlizers of reflexive pairs.*

Then G is monadic.

Proof of both theorems. The forward direction of 5.11 follows from theorem 5.8 part (ii) since T must preserve split coequalizers and so $G^\Pi : \mathcal{C}^\Pi \to \mathcal{C}$ creates G^Π-split coequalizers.

Now we will show 5.12 and the backwards direction of 5.11. We have $K : \mathcal{D} \to \mathcal{C}^\Pi$ where Π is the monad induced by $(F \dashv G)$. Define $L : \mathcal{C}^\Pi \to \mathcal{D}$ by setting $L(A, \alpha)$ to be the coequalizer of $F\alpha, \epsilon_{FA} : FGFA \to FA$ (note that this is reflexive since $F\eta_A$ is a common splittling, and G-split since

$$GFGFA \underset{\eta_{GFA}}{\overset{GF\alpha}{\underset{G\epsilon_{FA}}{\rightrightarrows}}\!\!\leftarrow} GFA \underset{h\alpha}{\overset{\alpha}{\rightleftarrows}} A$$

is a split coequealizer diagram). On morphisms L is defined so that

$$\begin{array}{ccccc} FGFA & \rightrightarrows & FA & \longrightarrow & L(A,\alpha) \\ {\scriptstyle FGFf}\downarrow & & \downarrow{\scriptstyle Ff} & & \downarrow{\scriptstyle Lf} \\ FGFB & \rightrightarrows & FB & \longrightarrow & L(B,\beta) \end{array}$$

commutes; this is clearly functorial. Note that

$$\begin{array}{ccccc} KFGFA & \underset{\mu_A}{\overset{GF\alpha}{\rightrightarrows}} & KFA & \xrightarrow{\alpha} & (A,\alpha) \\ & & & \searrow & \downarrow \\ & & & & KL(A,\alpha) \end{array}$$

is a G-split coequalizer so we get a unique factorization $(A, \alpha) \to KL(A, \alpha)$ which is natural in A. $KB = (GB, G\epsilon_B)$ so we have a coequalizer diagram

$$\begin{array}{ccccc} FGFGB & \underset{\epsilon_{FGB}}{\overset{FG\epsilon_B}{\rightrightarrows}} & FGB & \longrightarrow & LKB \\ & & & {\scriptstyle \epsilon_B}\searrow & \downarrow \\ & & & & B \end{array}$$

so we get a unique factorization $LKB \to B$ which is natural in B. The unit $(A, \alpha) \to KL(A, \alpha)$ maps to an isomorphism $A \to GL(A, \alpha)$ in $\mathcal{C}$ provided G preserves the coequalizer defining L, but G^Π reflects isomorphisms so it must be an isomorphism in $\mathcal{C}^\Pi$. Similarly, $LKB \to B$ maps to an isomorphism in $\mathcal{C}$, so if G reflects isomorphisms or if G creates the coequalizer of $FGFGB \rightrightarrows FGB$ then $KB \to B$ must be an isomorphism. □

Examples 5.13.

(a) For any category of algebras (in the universal algebra sense) e.g. **Gp**, **Rng**, **Mod**$_R$, the forgetful functor to **Set** is monadic. The left adjoint exists and the functor reflects isomorphisms. We note that if $A_1 \underset{g_1}{\overset{f_1}{\rightrightarrows}} B_1 \xrightarrow{h_1} C_1$ and $A_2 \underset{g_2}{\overset{f_2}{\rightrightarrows}} B_2 \xrightarrow{h_2} C_2$ are reflexive coequalizers in **Set** then

$$A_1 \times A_2 \underset{g_1 \times g_2}{\overset{f_1 \times f_2}{\rightrightarrows}} B_1 \times B_2 \xrightarrow{h_1 \times h_2} C_1 \times C_2$$

is a coequalizer: note that two elements $b_1, b_2 \in B_i$ are identified in C_i iff we can link them by a chain $b_1 c_1 c_2 \cdots c_n b_2$ where each adjascent pair is the image of either $(f, g) : A_i \to B_i \times B_i$ or $(g, f) : A_i \to B_i \times B_i$. If we have strings linking $b_{1,1}$ to $b_{1,2}$ and $b_{2,1}$ to $b_{2,2}$ we can link $(b_{1,1}, b_{2,1})$ to $(b_{1,2}, b_{2,1})$ to $(b_{1,2}, b_{2,2})$ since both pairs are reflexive. Hence if $A \rightrightarrows B \to C$ is a reflexive coequalizer in **Set** so is $A^n \rightrightarrows B^n \to C^n$ for any finite n. So if A and B have an n-ary operation and f, g are homomorphisms for $i = 1, 2$ we get a unique $C^n \to C$ making h a homomorphism. This shows that $U : \mathcal{A} \to \mathbf{Set}$ creates reflexive coequalizers.

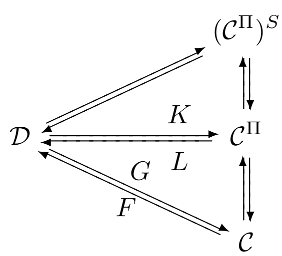
11/15/06

(b) Any reflection is monadic. The direct proof is on exercise sheet 3, but it can also be proved using theorem 5.11. Suppose $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ is a reflection: identify $\mathcal{D}$ with a full cubcategory of $\mathcal{C}$. If $f, g : A \to B$ is a G-split pair in $\mathcal{D}$ we have a split coequalizer diagram

$$A \underset{\underset{t}{\longleftarrow}}{\overset{f}{\underset{g}{\rightrightarrows}}} B \underset{s}{\overset{h}{\rightleftarrows}} C$$

in $\mathcal{C}$ and we need only show that $C \in \operatorname{ob} \mathcal{D}$. We know that $sh : B \to B$ is in $\mathcal{D}$ but $s : C \to B$ is an equalizer of sh and 1_B and $\mathcal{D}$ is closed under limits since its reflective in $\mathcal{C}$ so we see that C must be in $\mathcal{D}$ also.

(c) Consider the composite adjunction $\mathbf{Set} \underset{U}{\overset{F}{\rightleftarrows}} \mathbf{AbGp} \underset{I}{\overset{L}{\rightleftarrows}} \mathbf{tfAbGp}$ where $\mathbf{tfAbGp}$ is the category of torsion-free abelian groups. Each factor is monadic by the previous two examples, but the composite isn't since free abelian groups are torsion free and so the monat on $\mathbf{Set}$ induced by $(LF \dashv UI)$ is isomorphic to that induced by $(F \dashv U)$. In general, given an adjunction $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ where $\mathcal{D}$ has reflexive coequalizers we can form the "monadic tower"



where Π is the monad induced by $(F \dashv G)$, L is left adjoint to the comparison functor K, S is the monad induced by $(L \dashv K)$ and so on. We say $(F \dashv G)$ has *monadic length* n if this produces an equivalence after n steps. So $\mathbf{Set} \rightleftarrows \mathbf{TfAbGp}$ has monadic length 2.

(d) Consider the adjunction $D : \mathbf{Set} \rightleftarrows \mathbf{Top} : U$. The monad induced by this adjunction is $(1_{\mathbf{Set}}, 1, 1)$ so its category of algebras is isomorphic to $\mathbf{Set}$ and hence the adjunction has monadic length ∞.

(e) Consider the composite adjunction $\mathbf{Set} \underset{U}{\overset{D}{\rightleftarrows}} \mathbf{Top} \underset{I}{\overset{\beta}{\rightleftarrows}} \mathbf{kHaus}$. This *is* monadic. E. Moves gave a direct proof but we will use 5.11. We need to show that UI creates coequalizers of UI-split pairs. So suppose $f, g : X \to Y$ is a parallel pair in $\mathbf{kHaus}$ and

$$X \underset{\underset{t}{\longleftarrow}}{\overset{f}{\underset{g}{\rightrightarrows}}} Y \underset{s}{\overset{h}{\rightleftarrows}} Z$$

is a slit coequalizer diagram in $\mathbf{Set}$. We need to show there's a unique compact Hausdorff topology on Z which makes h continuous and that it's a coequalizer in $\mathbf{kHaus}$. We can think of Z as a quotient Y/R so if we equip it with the quotient topology we get a coequalizer in $\mathbf{Top}$. The quotient topology is certainly compact, so it's the only topology making h continuous which could possibly be Hausdorff. Fact: If Y is compact Hausdorff and $R \subseteq Y \times Y$ is an equivalence relation then Y/R is Hausdorff iff R is closed in $Y \times Y$. Claim: the equivalence relation R generated by $\{(f(x), g(x)) \,|\, x \in X\}$ is the set $\{(g(x_1), g(x_2)) \,|\, x_1, x_2 \in X \text{ s.t. } f(x_1) = f(x_2)\}$. For if $(y_1, y_2) \in R$ then $h(y_1) = h(y_2)$ so $ft(y_1) = sh(y_1) = sh(y_2) = ft(y_2)$ so $y_1 = g(x_1)$, $y_2 = g(x_2)$ where $x_i = t(y_i)$ and $f(x_1) = f(x_2)$. The set above is closed in $X \times X$ since Y is Hausdorff. Thus Y/R is compact, and its image under $g \times g$ is compact and hence closed in $Y \times Y$.

6. Abelian Categories

Definition 6.1. Let $\mathcal{A}$ be a category equipped with a forgetful functor $U : \mathcal{A} \to \mathbf{Set}$. We say a locally small category $\mathcal{C}$ is *enriched over* $\mathcal{A}$ if we're given a factorization of $\mathcal{C}(-,-) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$ through U. If $\mathcal{A} = \mathbf{Set}_*$ we say $\mathcal{C}$ is a *pointed* category. If $\mathcal{A} = \mathbf{CMon}$ we say $\mathcal{C}$ is *semi-additive*. If $\mathcal{A} = \mathbf{AbGp}$ then we say $\mathcal{C}$ is *additive*.

Lemma 6.2.

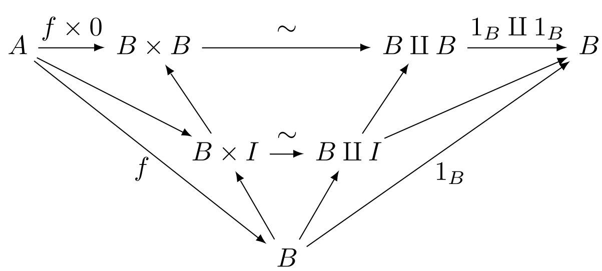
(i) *If $\mathcal{C}$ is pointed and $I \in \mathrm{ob}\,\mathcal{C}$ the following are equivalent:*
   - (a) *I is initial*
   - (b) *I is terminal*
   - (c) $1_I = 0 : I \to I$.

(ii) *If $\mathcal{C}$ is semi-additive and $A, B, C \in \mathrm{ob}\,\mathcal{C}$ the following are equivalent:*
   - (a) *There exist $\pi_1 : C \to A$ and $\pi_2 : C \to B$ making C a product $A \times B$.*
   - (b) *There exist $\nu_1 : A \to C$ and $\nu_2 : B \to C$ making C a coproduct $A \amalg B$.*
   - (c) *There exist morphisms $\pi_1, \pi_2, \nu_1, \nu_2$ (as above) satisfying $\pi_1\nu_1 = 1_A$, $\pi_2\nu_2 = 1_B$, $\pi_2\nu_1 = 0$, $\pi_1\nu_2 = 0$ and $\nu_1\pi_2 + \nu_2\pi_2 = 1_C$.*

The proof is left as an exercise.

Lemma 6.3. *Suppose $\mathcal{C}$ is a locally small category with finite products and coproducts such that $0 : \emptyset \to *$ is an isomorphism and the morphism $A \amalg B \to A \times B$ (induced by 1_A and 1_B), is an isomorphism. Then $\mathcal{C}$ has a unique semi-additive structure where $0 : A \to B$ is the unique morphism factoring through 0.*

Proof. The 0 of the semi-additive structure has to be as defined as in the statement, since we need $0f = g0 = 0$ for all f and g. Given $f, g : A \to B$ we define $f +_\ell g$ to be $A \xrightarrow{f \times g} B \times B \xrightarrow{\sim} B \amalg B \xrightarrow{1_B \amalg 1_B} B$ and $f +_r g$ to be $A \xrightarrow{1_A \times 1_A} A \times A \xrightarrow{\sim} A \amalg A \xrightarrow{f \amalg g} B$. We claim that 0 is a unit for both $+_\ell$ and $+_r$. Consider $f +_\ell 0$, and consider the following diagram which shows the desired statement:



Given four morphisms $f, g, h, k : A \to B$ consider

$$
\begin{aligned}
&(f +_\ell g) +_r (h +_\ell k) = \\
&\quad = A \xrightarrow{1 \times 1} A \times A \xrightarrow{\sim} A \amalg A \xrightarrow{(f \times h) \amalg (g \times k)} B \times B \xrightarrow{\sim} B \amalg B \xrightarrow{1 \amalg 1} B \\
&\quad = A \xrightarrow{1 \times 1} A \times A \xrightarrow{\sim} A \amalg A \xrightarrow{(f +_\ell h) \amalg (g +_\ell k)} B \\
&\quad = (f +_r g) +_\ell (h +_r k)
\end{aligned}
$$

so $+_\ell = +_r$ and it is an associative and commutative operation.

11/17/07

For the uniqueness, recall from the previous lemma that if we have any semi-additive structure then the identity map $A \times A \to A \times A$ is equal to $\nu_1\pi_1 + \nu_2\pi_2$. So given $f, g : A \to B$ the composite

$$
\begin{aligned}
A &\xrightarrow{1 \times 1} A \times A \xrightarrow{\sim} A \amalg A \xrightarrow{f \amalg g} B = \\
&= A \xrightarrow{1 \times 1} A \times A \xrightarrow{\nu_1\pi_1 + \nu_2\pi_2} A \times A \xrightarrow{\sim} A \amalg A \xrightarrow{f \amalg g} B \\
&= A \xrightarrow{\nu_1 + \nu_2} A \amalg A \xrightarrow{f \amalg g} B = A \xrightarrow{f + g} B
\end{aligned}
$$

Thus $f + g = f +_r g$ and the structure is unique. $\square$

Definition 6.4. An object which is both inital and terminal is called a *zero object*. An object which is both a product $A \times B$ and a coproduct $A \amalg B$ is called a *biproduct* and denoted $A \oplus B$. We will use product notation for maps between biproducts.

Corollary 6.5. *Let $\mathcal{C}$ and $\mathcal{D}$ be semi-additive categories with finite products. The functor $F : \mathcal{C} \to \mathcal{D}$ preserves finite products iff it preserves addition, i.e. iff $F(0) = 0$ and $F(f + g) = Ff + Fg$.*

Proof. If F preserves addition then it preserves biproducts by lemma 6.2. The converse follows from lemma 6.3. □

Definition 6.6. Let $\mathcal{C}$ be a pointed category. By a *kernel* (dually, a *cokernel*) of a morphism $f : A \to B$ we mean an equalizer (dually, a coequalizer) of f and 0 We say a monomorphism (dually, an epimorphism) is *normal* if it occurs as a kernel (cokernel). We say $f : A \to B$ is a *pseudo-epimorphism* if $fg = 0$ implies $g = 0$ (equivalently, the kernel of f is $0 \to A$).

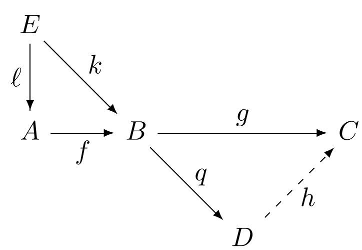
If $\mathcal{C}$ is additive then every regular monomorphism is normal, since the equalizer of $f, g : A \to B$ has the same univeral property as the kernel of $f - g$. And every pseudo-morphism is monic since $fg = fh$ iff $f(g - h) = 0$.

In **Gp** every monomorphism is regular, but a monomorphism $H \to G$ is normal iff H is a normal subgroup of G. But every epimorphism $f : G \to K$ is normal, since if f is surjective then $K \cong G/\ker f$.

In **Set** every monomorphism is normal, since if $f : A \to B$ is injective it's the kernel of $B \to B/\sim$ where $b_1 \sim b_2$ iff $b_1 = b_2$ or $\{b_1, b_2\} \subset \mathrm{im} f$. But not every epimorphism in $\mathbf{Set}_*$ is normal.

Lemma 6.7. *Let $\mathcal{C}$ be a pointed category with cokernels. Then $f : A \to B$ is a normal monomorphism iff $f = \ker \operatorname{coker} f$.*

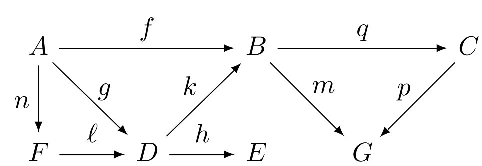
Proof. The backwards direction is trivial. For the forwards direction, suppose $f = \ker(g : B \to C)$. Let $q = \operatorname{coker} f$. Then g factors as hq since $gf = 0$. Now given any $k : E \to B$ with $qk = 0$ we have $gk = hqk = 0$ so there's a unique factorization $k = f\ell$. Thus any k such that $qk = 0$ factors through f and so $f = \ker q = \ker \operatorname{coker} f$.



□

Lemma 6.8. *Suppose $\mathcal{C}$ is pointed with kernels and cokernels and every monomorphism in $\mathcal{C}$ is normal. Then every morphism of $\mathcal{C}$ factors as a pseudo-epimorphism followed by a monomorphism, and the factorization is unique up to isomorphism.*

Proof. Given $f : A \to B$, let $q : B \to C$ be the cokernel of f and let $k : D \to B$ be the kernel of q. We get a factorization $f = kg$; we claim g is pseudo-epic. Supose $h : D \to E$ satisfies $hg = 0$ and let $\ell = \ker h$. Then $k\ell$ is monic so $k\ell = \ker m$ for some m. We can factor g as ℓn so $f = kg = k\ell n$, so $mf = 0$, so $m = pq$ for some p. Now $qk = 0$ since $k = \ker q$ so $mk = 0$ so k factors through $k\ell$. But k and ℓ are monic so this forces ℓ to be an isomorphism and hence $h = 0$.



For uniqueness, suppose f factors as kg where g is pseudo-epic. Then $\operatorname{coker} f = \operatorname{coker} k$. So if k is also a monomorphism then $g = \ker \operatorname{coker} k = \ker \operatorname{coker} f$ by 6.7. □

Definition 6.9. An *abelian category* is an additive category with finite limits and colimits (equivalently finite coproducts and products, kernels and cokernels) in which every monomorphism and every epimorphism is regular (equivalently, normal).

Example 6.10. **AbGp**, $\mathbf{Mod}_R$, $[\mathcal{C}, \mathcal{A}]$ where $\mathcal{A}$ is abelian. If $\mathcal{C}$ is additive and $\mathcal{A}$ is abelian then the subcategory $\mathbf{Add}(\mathcal{C}, \mathcal{A}) \subseteq [\mathcal{C}, \mathcal{A}]$ of additive functors $\mathcal{C} \to \mathcal{A}$ is abelian. Note that $\mathbf{Mod}_R = \mathbf{Add}(R, \mathbf{AbGp})$ where we consider a ring R as an additive category with one object.

11/20/06

In a pointed category with kernels and cokernels we write $\operatorname{im} f$ for $\ker \operatorname{coker} f$ and $\operatorname{coim} f$ for $\operatorname{coker} \ker f$. In an abelian category, any f factors as $(\operatorname{im} f)g$ with g epic, and as $h(\operatorname{coim} f)$ with h monic (by 6.8) and these factorizations must be isomorphic. In general, we get a comparison map

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle \operatorname{coim} f}\downarrow & & \uparrow{\scriptstyle \operatorname{im} f} \\ E & \xrightarrow{\overline{f}} & D \end{array}$$

and in an abelian category $\overline{f}$ is always an isomorphism.

Note that $\mathcal{A}$ is abelian iff $\mathcal{A}$ is additive with finite limits and colimits and every f factors as $(\operatorname{im} f)(\operatorname{coim} f)$.

Lemma 6.11. *Suppose we are given a pullback square*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ C & \xrightarrow{k} & D \end{array}$$

in an abelian category with h epic. Then the square is also a pushout and g is epic.

Proof. Consider the diagram $A \xrightarrow{f \times -g} B \oplus C \xrightarrow{h \amalg k} D$. We have $(h \amalg k)(f \times -g) = hf - kg = 0$ and the fact that (f, g) has the universal property of a pullback implies that $f \times -g = \ker(h \amalg k)$. But $(h \amalg k)(1 \times 0) = h$ is epic so $h \amalg k$ is epic and therefore by 6.7 $h \amalg k = \operatorname{coker}(f \times -g)$, so the original square is a pushout.

Now consider the cokernel $\epsilon : C \to E$ of g. Then ϵ and $0 : B \to E$ form a cone under $C \xleftarrow{g} A \xrightarrow{f} B$ so they factor uniquely through D, say by $r : D \to E$. Then $rh = 0$ but h is epic so $r = 0$ and therefore $q = rk = 0$. Hence g is an epimorphism. ☐

Definition 6.12. We say a sequence of morphisms $\cdots \longrightarrow A \xrightarrow{g} B \xrightarrow{f} C \longrightarrow \cdots$ is *exact* at B if $\ker f = \operatorname{im} g$ (or, equivalently, $\operatorname{coker} g = \operatorname{coim} f$). Note that $f : A \to B$ is monic iff $0 \longrightarrow A \xrightarrow{f} B$ is exact, and $f : A \to B$ is epic iff $A \xrightarrow{f} B \longrightarrow 0$ is exact. A functor $F : \mathcal{A} \to \mathcal{B}$ between abelian categories is called *exact* if it preserves exactness of sequences. We say F is *left exact* if it preserves exactness of sequences of the form $0 \longrightarrow A \longrightarrow B \longrightarrow C$, and F is *right exact* if it preserves exactness of sequences of the form $A \longrightarrow B \longrightarrow C \longrightarrow 0$.

By considering the exact sequences

$$0 \longrightarrow A \xrightarrow{1 \times 0} A \oplus B \xrightarrow{0 \amalg 1} B \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow B \xrightarrow{0 \times 1} A \oplus B \xrightarrow{1 \amalg 0} A \longrightarrow 0$$

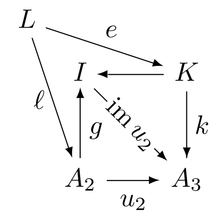
we see that any left exact functor must preserve biproducts, i.e. it must be additive. Hence F is left exact iff F preserves all finite limits. Also, F is exact iff F preserves kernels and cokernels iff F preserves all finite limits and colimits.

Lemma 6.13 (Five Lemma)**.** *Suppose we are given a diagram*

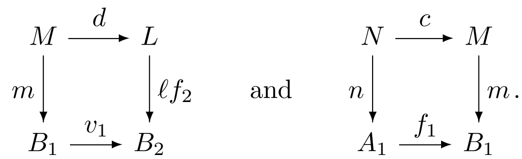
$$\begin{array}{ccccccccc} A_1 & \xrightarrow{u_1} & A_2 & \xrightarrow{u_2} & A_3 & \xrightarrow{u_3} & A_4 & \xrightarrow{u_4} & A_5 \\ {\scriptstyle f_1}\downarrow & & {\scriptstyle f_2}\downarrow & & {\scriptstyle f_3}\downarrow & & {\scriptstyle f_4}\downarrow & & {\scriptstyle f_5}\downarrow \\ B_1 & \xrightarrow{v_1} & B_2 & \xrightarrow{v_2} & B_3 & \xrightarrow{v_3} & B_4 & \xrightarrow{v_4} & B_5 \end{array}$$

in an abelian category where the rows are exact. Suppose also that f_1 is epic, f_2 and f_4 are isomorphisms and f_5 is monic. Then f_3 is an isomorphism.

Proof. First we show that f_3 is monic. Let $k : K \to A_3$ be the kernel of f_3. Now $f_4 u_3 k = v_3 f_3 k = 0$ and f_4 is monic so $u_e k = 0$, so k factors through $\ker u_3 = \operatorname{im} u_2$. Hence if L is the pullback of k and u_2 in

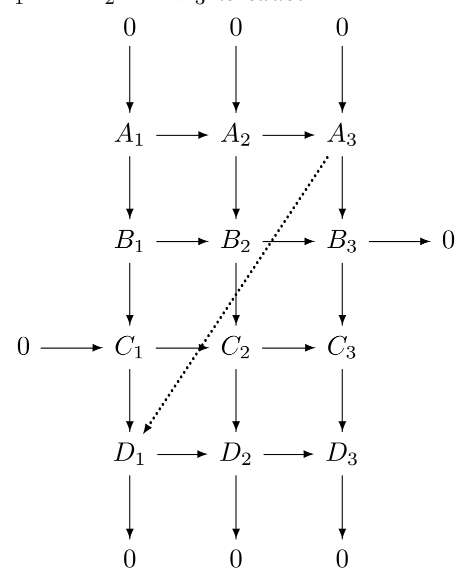


it is isomorphic to the pullback of $A_2 \longrightarrow I \longleftarrow K$, so $e : L \to K$ is epic (as g is epic). Now $v_2 f_2 \ell = f_3 u_2 \ell = f_3 k e = 0$ so $f_2 \ell$ factors through $\ker v_2 = \operatorname{im} v_1$. Consider the pullbacks



Then d is epic (by the same argument as above) and c is epic (as f_1 is epic). $f_2 \ell d c = v_1 m c = v_1 f_1 n = f_2 u_1 n$; f_2 is monic so $\ell d c = u_1 n$. Now $kedc = u_2 \ell d c = u_2 u_1 n = 0$. But edc is epic so $k = 0$, i.e. f_3 is monic. Dually, f_3 is epic, so it is an isomorphism. □

Lemma 6.14 (Snake Lemma)**.** *Suppose we are given a diagram as below, in which the columns are exact, the two middle rows are exact, and all of the squares commute. Then there exists a morphism $A_3 \to D_1$ such that $A_1 \longrightarrow A_2 \longrightarrow A_3 \longrightarrow D_1 \longrightarrow D_2 \longrightarrow D_3$ is exact.*



The proof is omitted.

11/22/06

Definition 6.15. By a *complex* in an abelian category $\mathcal{A}$ we mean a sequence

$$\cdots \longrightarrow C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \longrightarrow \cdots$$

of objects and morphisms such that $d_n d_{n+1} = 0$ for all n. Note that this is just an additive functor $Z \to \mathcal{A}$ where $\operatorname{ob} Z = \mathbb{Z}$, $Z(n,n) = \mathbb{Z}$ (with 1 as the identity morphism), $Z(n, n-1) = \mathbb{Z}$, and $Z(n,m) = \{0\}$ if $m \neq n, n-1$ (with the obvious definition of composition). Hence the complexes of $\mathcal{A}$ are the objects of an abelian category $c\mathcal{A} = \mathbf{Add}(Z, \mathcal{A})$. Given a complex C we define $Z_n \to C_n$ to be the kernel of $C_n \to C_{n-1}$, $B_n \to C_n : \operatorname{im}(d_{n+1})$, $Z_n \to H_n = \operatorname{coker}(B_n \to Z_n)$. Equivalently, we could form $C_n \to A_n = \operatorname{coker}(d_{n+1})$

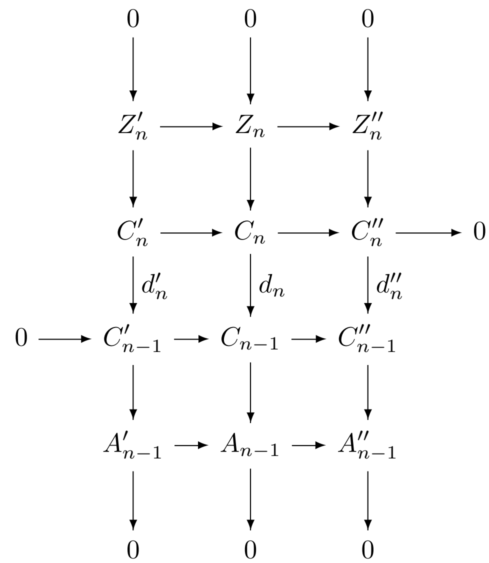
and then $Z_n \to H_n \to A_n$ is the image factorization of $Z_n \to C_n \to A_n$. Each of $(C_* \mapsto Z_n)$, $(C_* \mapsto A_n)$, $(C_* \mapsto B_n)$ and $(C_* \mapsto H_n)$ defines an additive functor $c\mathcal{A} \to \mathcal{A}$. Note that $H_n = 0$ iff C_* is exact at C_n.

Theorem 6.16 (Mayer-Vietoris)**.** *Suppose that we are given an exact sequence $0 \to C'_* \to C_* \to C''_* \to 0$ in $c\mathcal{A}$. Then there is an exact sequence*

$$\cdots \longrightarrow H'_n \longrightarrow H_n \longrightarrow H''_n \longrightarrow H'_{n-1} \longrightarrow H_{n-1} \longrightarrow \cdots$$

of homology objects in $\mathcal{A}$.

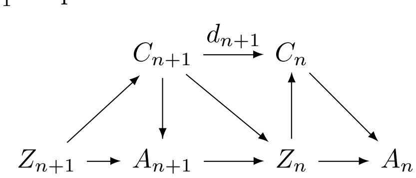
Proof. First consider the diagram



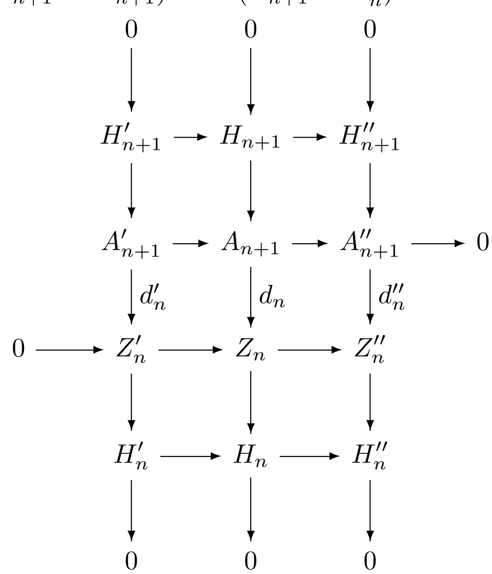
By lemma 6.14 the top and bottom rows are exact. Moreover $Z'_n \to Z_n$ is monic since

$$Z'_n \to Z_n \to C_n = Z'_n \to C'_n \to C_n$$

is monic and similarly $A_{n-1} \to A''_{n-1}$ is epic. Now consider



Note that $H_{n+1} \to A_{n+1} = \mathrm{im}\,(Z_{n+1} \to A_{n+1}) = \ker(A_{n+1} \to Z'_n)$. Now we can consider



By 6.14 we get a morphism $H''_{n+1} \to H'_n$ making the sequence $H'_{n+1} \to H_{n+1} \to H''_{n+1} \to H'_n \to H_n \to H''_n$ exact. □

7. Monoidal and Closed Categories

We frequently encounter instances of a category $\mathcal{C}$ equipped with a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and an object $I \in \operatorname{ob} \mathcal{C}$ which makes $\mathcal{C}$ into a monoid up to isomorphism in **Cat**.

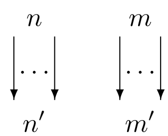
Examples 7.1.

(a) Any category with finite products, with $\otimes = \times$ and $I = *$. We know that $A \times (B \times C) \cong (A \times B) \times C$ and $* \times A \cong A \cong A \times *$ since they are limits of the same diagrams. Similarly, any any category with finite coproducts with $\otimes = \amalg$ and $I = \emptyset$.

(b) In **AbGp** we have the usual tensor product $\otimes$ with unit $\mathbb{Z}$. In $\mathbf{Mod}_R$ (for R commutative) we have $\otimes_R$ with unit R.

(c) For any $\mathcal{C}$ we have a monoidal structure on $[\mathcal{C}, \mathcal{C}]$ where $\otimes$ is composition of functors and I is the identity functor.

(d) Consider the category Δ with $\operatorname{ob} \Delta = \mathcal{N}$ and morphisms $n \to m$ are order preserving maps $\{0, \ldots, n-1\} \to \{0, \ldots, m-1\}$. This has a monoidal structure given on objects by $+$ and on morphisms combining maps in parallel (?) e.g. $n + m \xrightarrow{+} n' + m'$ by



Note that although $n + m = m + n$ this isn't a natural isomorphism.

Definition 7.2. By a *monoidal structure* on a category $\mathcal{C}$ we mean a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and an object I equipped with natural isomorphisms $\alpha_{A,B,C} : A \otimes (B \otimes C) \to (A \otimes B) \otimes C$, $\lambda_A : I \otimes A \to A$ and $\rho_A : A \otimes I \to A$ such that all diagrams constructed from instances of α, λ, ρ commute. In particular, we ask that the diagrams

$$\begin{array}{ccccc} A \otimes (B \otimes (C \otimes D)) & & \xrightarrow{\alpha_{A,B,C\otimes D}} & & (A \otimes B) \otimes (C \otimes D) \\ \downarrow{\scriptstyle 1_A \otimes \alpha_{B,C,D}} & & & & \downarrow{\scriptstyle \alpha_{A\otimes B,C,D}} \\ A \otimes ((B \otimes C) \otimes D) & \xrightarrow[\alpha_{A,B\otimes C,D}]{} & (A \otimes (B \otimes C)) \otimes D & \xrightarrow{\alpha_{A,B,C} \otimes 1_D} & ((A \otimes B) \otimes C) \otimes D \end{array}$$

and

$$\begin{array}{ccc} A \otimes (I \otimes B) & \xrightarrow{\alpha_{A,I,B}} & (A \otimes I) \otimes B \\ & {\scriptstyle 1_A \otimes \lambda_B} \searrow \quad \swarrow {\scriptstyle \rho_A \otimes 1_B} & \\ & A \otimes B & \end{array}$$

commute. Note that for $(\mathbf{AbGp}, \otimes, \mathbb{Z})$ the usual α sends a generator $a \otimes (b \otimes c)$ to $(a \otimes b) \otimes c$, but we also have an isomorphism $\overline{\alpha}$ sending $a \otimes (b \otimes c)$ to $-(a \otimes b) \otimes c$, but this doesn't satisfy the pentagon condition.

Theorem 7.3 (Coherence Theorem for Monoidal Categories)**.** *If these two diagrams commute then everything does. More formally, we define a set of words in $\otimes$ and I as follows: we have a stack of variables $A, B, C, D, \ldots$ which are words, I is a word, if u and v are words then $(u \otimes v)$ is a word. If u, v, w are words then $\alpha_{u,v,w} : u \otimes (v \otimes w) \to (u \otimes v) \otimes w$ is an instance of α (similarly an instance of λ and ρ). Also, if $\theta : v \to v'$ is an instance of α, λ or ρ so are $1_u \otimes \theta : (u \otimes v) \to (u \otimes v')$ and $\theta \otimes 1_w : (v \otimes w) \to (v' \otimes w)$. The body of a word is the sequence of variables that appears in it. The theorem says: given two words w, w' with the same body there is a unique isomorphism $w \to w'$ obtainable by composing instsances of α, λ, ρ and their inverses.*

11/24/06

Proof. Note that a word involving n variables defines a functor $\mathcal{C}^n \to \mathcal{C}$ and each instance of α, λ, or ρ defines a natural isomorphism between two such functors. We define a *reduction step* to be an instance of α, λ or ρ (as opposed to their inverses). We define the *height* $h(w)$ of a word to be $a(w) + i(w)$, where $i(w)$ is the number of occurrences of I in w and $a(w)$ is the number instances of a $\otimes$ occurring before a (. Note that if $\theta : w \to w'$ is an instance of α then $i(w) = i(w')$ and $a(w) > a(w')$, and if θ is an instance of λ or ρ then $i(w) > i(w')$ and $a(w) \geq a(w')$. Hence any sequence of reduction steps starting from w must terminate at a *reduced word* from which no further reductions are possible. Reduced words are those of height 0: $(\cdots((A_1 \otimes A_2) \otimes A_3) \otimes \cdots) \otimes A_n$ and the word I of height 1. These are the only reduced words, since if $i(w) > 0$ and $w \neq I$ then w has a subword $(y \otimes I)$ or $(I \otimes v)$ to which we can apply ρ or λ. If $a(w) > 0$ then there is a substring $\cdots \otimes (\cdot$ in w and hence a subword $(u \otimes (v \otimes x))$ to which we can apply α. For any w any reduction path from w must lead to a reduced word w_0 with the same body.

Note that in order to prove the theorem it suffices to show that any sequence of reduction steps can be put into a commutative diagram. In particular, if we can show that there is a unique morphism $\theta_w : w \to w_0$ then any morphism $w \to w'$ which is a composition of α, ρ, λ's (and their inverses) must be a composite $\theta_{w'}^{-1}\theta_w$, so any two of these can be put into a commutative diagram.

To prove that any pair of reduction steps θ, ϕ can be embedded in a commutative polygon we consider the following cases.

Case 1: θ and ϕ operate on disjoint subwords. So $w = \cdots(v \otimes w)\cdots$ and $\theta = \cdots(\theta' \otimes 1)\cdots$ and $\phi = \cdots(1 \otimes \phi')\cdots$. Then we have the following diagram

$$\begin{array}{ccc} (u \otimes v) & \xrightarrow{1 \otimes \phi'} & (u \otimes v') \\ {\scriptstyle \theta' \otimes 1}\downarrow & & \downarrow{\scriptstyle \theta' \otimes 1} \\ (u' \otimes v) & \xrightarrow{1 \otimes \phi'} & (u' \otimes v') \end{array}$$

by functoriality of $\otimes$.

Case 2: ϕ operates within one argument of θ, e.g. $\theta = \alpha_{u,v,x} : u \otimes (v \otimes x) \to (u \otimes v) \otimes x$ and $\phi = (1 \otimes (\phi' \otimes 1))$ where $\phi' : v \to v'$. Then we have

$$\begin{array}{ccc} u \otimes (v \otimes x) & \xrightarrow{1 \otimes (\phi' \otimes 1)} & u \otimes (v' \otimes x) \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \alpha} \\ (u \otimes v) \otimes x & \xrightarrow{(1 \otimes \phi') \otimes 1} & (u \otimes v') \otimes x \end{array}$$

by naturality of α.

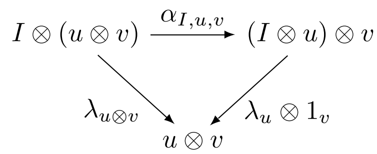
Case 3: θ and ϕ interfere with each other.

If θ, ϕ are both α's w must contain a subword $u \otimes (v \otimes (x \otimes y))$ and θ, ϕ are $\alpha_{u,v,x \otimes y}$ and $1 \otimes \alpha_{v,x,y}$ in some order. Then we simply use the pentagon identity. If θ is a λ and ϕ is a ρ then w contains $\cdots I \otimes I \cdots$ and $\theta = \lambda_I$, $\phi = \rho_I$ so we need to know that $\lambda_I = \rho_I$. To see this note that

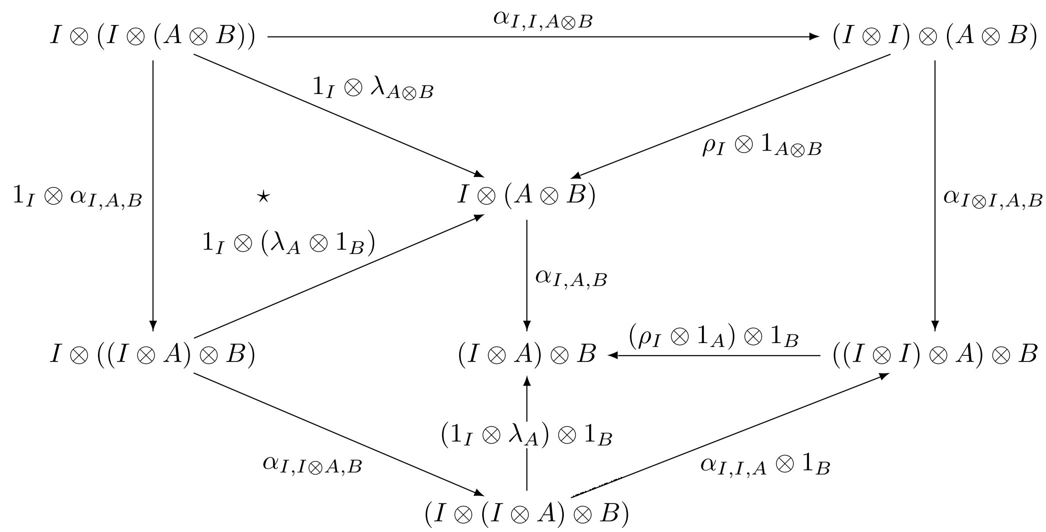
$$\begin{array}{ccccc} & & I \otimes I & & \\ & \nearrow^{\lambda_{I \otimes I}} & & \nwarrow^{\lambda_I \otimes 1_I} & \\ I \otimes (I \otimes I) & & \xrightarrow{\alpha_{I,I,I}} & & (I \otimes I) \otimes I \\ & \searrow_{1_I \otimes \lambda_I} & & \swarrow_{\rho_I \otimes 1_I} & \\ & & I \otimes I & & \end{array}$$

commutes. But $1_I \otimes \lambda_I = \lambda_{I \otimes I}$ as $\lambda_I(1_I \otimes \lambda_I) = \lambda_I \lambda_{I \otimes I}$ by naturality of λ and λ_I is an isomorphism. Since $\alpha_{I,I,I}$ is also an isomorphism it follows that $\rho_I \otimes 1_I = \lambda_I \otimes 1_I$. But $\cdot \otimes I$ is naturally isomorphic to the identity so $\rho_I = \lambda_I$.

If θ is an α and ϕ is a λ then either w contains $u \otimes (I \otimes v)$, $\theta = \alpha_{u,I,v}$ and $\phi = 1_u \otimes \lambda_v$ (so we can use the triangle) or w contains $I \otimes (u \otimes v)$, $\theta = \alpha_{I,u,v}$ and $\phi = \lambda_{u \otimes v}$. For this case we need to know that



commutes. Note that it suffices to prove this for this triangle with a leading $I\otimes$ added, since $I \otimes \cdot$ is naturally isomorphic to the identity. Thus what we want to show is that triangle $\star$ in the following diagram commutes:

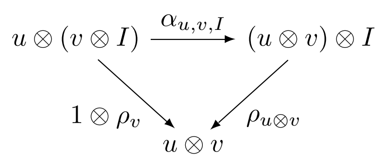


Note that the outside of this diagram is an instance of the α-pentagon. The two unlabelled triangles are instances of the α-λ-ρ identity, and the two quadrilateral cells commute by naturality of α. But from this we see that

$$\alpha_{I,A,B}(1_I \otimes \lambda_{A\otimes B}) = \alpha_{I,A,B}(1_I \otimes (\lambda_A \otimes 1_B))(1_I \otimes \alpha_{I,A,B})$$

and as $\alpha_{I,A,B}$ is an isomorphism triangle $\star$ also commutes.

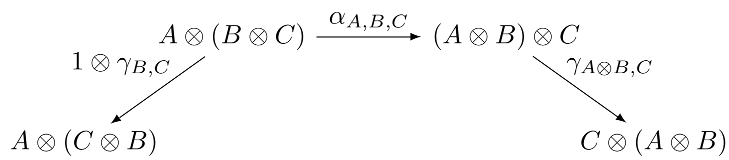
If θ is an α and ϕ is a ρ then w contains $u \otimes (v \otimes I)$, $\theta = \alpha_{u,v,I}$ $\phi = I \otimes \rho_v$ so we need to know that



commutes. This is shown analogously to the proof above using the pentagon between $A\otimes(B\otimes(I\otimes I))$ and $((A \otimes B) \otimes I) \otimes I$ and the fact that all of the maps in the pentagon are isomorphisms.

□

Definition 7.4. Let $(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ be a monoidal category. By a *symmetry* for $\otimes$ we mean a natural transformation $\gamma_{A,B} : A \otimes B \to B \otimes A$ satisfying



and

$$\begin{array}{ccc} I\otimes A & \xrightarrow{\gamma_{I,A}} & A\otimes I \\ & {\scriptstyle \lambda_A}\searrow \quad \swarrow{\scriptstyle \rho_A} & \\ & A & \end{array} \qquad \begin{array}{ccc} A\otimes B & \xrightarrow{\gamma_{A,B}} & B\otimes A \\ & {\scriptstyle 1_{A\otimes B}}\searrow \quad \swarrow{\scriptstyle \gamma_{B,A}} & \\ & A\otimes B & \end{array}$$

There is a coherence theorem for symmetric monoidal categories similar to 7.3 (but more delicate: note that $\gamma_{A,A} \neq 1_{A\otimes A}$ in general).

Warning: a given monoidal category may have more than one symmetry. For example, take $\mathcal{C} = \mathbf{AbGp}^{\mathbb{Z}}$ with $(A_* \otimes B_*)_n = \bigotimes_{p+q=n} A_p \otimes B_q$ and $I_n = \mathbb{Z}$ for $n = 0$ and 0 otherwise. We could define $\gamma_{A,B}$ to be the map $a\otimes b \mapsto b\otimes a$ or we could take $a \otimes b \mapsto (-1)^{pq} b\otimes a$ where $a \in A_p$ and $b \in B_q$. Both of these satisfy the above conditions.

11/27/06

Definition 7.5. Let $\mathcal{C}$ and $\mathcal{D}$ be monoidal categories, and $F : \mathcal{C} \to \mathcal{D}$ a functor. By a (*lax*) *monoidal structure* on F we mean a natural transformation $\theta_{A,B} : FA\otimes FB \to F(A\otimes B)$ and a morphism $c : I \to FI$ such that the diagrams

$$\begin{array}{ccccc} FA\otimes(FB\otimes FC) & \xrightarrow{1\otimes\theta_{B,C}} & FA\otimes F(B\otimes C) & \xrightarrow{\theta_{A,B\otimes C}} & F(A\otimes(B\otimes C)) \\ \downarrow{\scriptstyle \alpha_{FA,FB,FC}} & & & & \downarrow{\scriptstyle F\alpha_{A,B,C}} \\ (FA\otimes FB)\otimes FC & \xrightarrow{\theta_{A,B}\otimes 1} & F(A\otimes B)\otimes FC & \xrightarrow{\theta_{A\otimes B,C}} & F((A\otimes B)\otimes C) \end{array}$$

and

$$\begin{array}{ccc} I\otimes FA & \xrightarrow{c\otimes 1} & FI\otimes FA \\ \downarrow{\scriptstyle \lambda_{FA}} & & \downarrow{\scriptstyle \theta_{I,A}} \\ FA & \xrightarrow{F\lambda_A} & F(I\otimes A) \end{array}$$

and the analogous diagram for ρ commute. If the monoidal structures on $\mathcal{C}$ and $\mathcal{D}$ are symmetric we say that (θ, c) is a *symmetric* monoidal structure if

$$\begin{array}{ccc} FA\otimes FB & \xrightarrow{\theta_{A,B}} & F(A\otimes B) \\ \downarrow{\scriptstyle \gamma_{FA,FB}} & & \downarrow{\scriptstyle F\gamma_{A,B}} \\ FB\otimes FA & \xrightarrow{\theta_{B,A}} & F(B\otimes A) \end{array}$$

commutes. We say that (θ, c) is a *strong monoidal structure* if θ and c are isomorphisms. Given monoidal functors (F, θ, c) and (G, γ, k) we say a natural transformation $\beta : F \to G$ is *monoidal* if

$$\begin{array}{ccc} FA\otimes FB & \xrightarrow{\theta} & F(A\otimes B) \\ \downarrow{\scriptstyle \beta_A\otimes\beta_B} & & \downarrow{\scriptstyle \beta_{A\otimes B}} \\ GA\otimes GB & \xrightarrow{\gamma} & G(A\otimes B) \end{array} \qquad \text{and} \qquad \begin{array}{ccc} I & \xrightarrow{c} & FI \\ & {\scriptstyle k}\searrow & \downarrow{\scriptstyle \beta_I} \\ & & GI \end{array}$$

commute.

Examples 7.6.

(a) Let R be a commutative ring. The forgetful functor $(\mathbf{Mod}_R, \otimes_R, R) \to (\mathbf{AbGp}, \otimes, \mathbb{Z})$ is lax monoidal: if A and B are R-modules we have a quotient map $A\otimes B \to A\otimes_R B$ and $i : \mathbb{Z} \to R$ sending n to $n \cdot 1_R$.

(b) The forgetful functor $(\mathbf{AbGp}, \otimes, \mathbb{Z}) \to (\mathbf{Set}, \times, 1)$ is lax monoidal: we take the universal bilinear map $A \times B \to A\otimes B$ where $(a, b) \mapsto a\otimes b$ for $\otimes$ and $i : 1 \to \mathbb{Z}$ picks out the generator $1 \in \mathbb{Z}$.

(c) The functor $\mathbf{AbGp} \to \mathbf{Mod}_R$ which sends A to $R \otimes A$ is strong monoidal: we have canonical isomorphisms $R \otimes \mathbb{Z} \cong R$ and $(R \otimes A) \otimes_R (R \otimes B) \cong R \otimes (A \otimes_R R) \otimes B \cong R \otimes (A \otimes B)$. In general given a monoidal adjunction $(F \dashv G)$ (i.e. one for which the unit and counit are monoidal natural transformations) between lax monoidal functors the left adjoint is always strong: we get an inverse for $FA \otimes FB \to F(A \otimes B)$ from the composite

$$F(A \otimes B) \xrightarrow{F(\eta_A \otimes \eta_B)} F(GFA \otimes GFB) \longrightarrow FG(FA \otimes FB) \xrightarrow{\epsilon_{FA \otimes FB}} FA \otimes FB$$

(d) If $(\mathcal{C}, \times, 1)$ and $(\mathcal{D}, \times, 1)$ are cartesian monoidal categories then $F : \mathcal{C} \to \mathcal{D}$ is strong monoidal iff F preserves finite products.

$$\begin{array}{ccc} FA \times FB & \xrightarrow{\theta_{A,B}} & F(A \times B) \\ \downarrow & & \downarrow \\ FA \times F1 & \longrightarrow & F(A \times 1) \\ \uparrow & & \uparrow \\ FA \times 1 & \longrightarrow & FA \end{array}$$

shows that θ commutes with the projections.

(e) Any functor F between cocartesian monoidal categories has a unique lax monoidal structure and this structure is strong iff F preserves finite coproducts.

Definition 7.7. Let $(\mathcal{C}, \otimes, I)$ be a monoidal category. By a *monoid* in $\mathcal{C}$ we mean an object A equipped with morphisms $m : A \otimes A \to A$ and $e : I \to A$ such that

$$\begin{array}{ccccc} A \otimes (A \otimes A) & & \xrightarrow{1 \otimes m} & & A \otimes A \\ \downarrow{\scriptstyle \alpha_{A,A,A}} & & & & \downarrow{\scriptstyle m} \\ (A \otimes A) \otimes A & \xrightarrow{m \otimes 1} & A \otimes A & \xrightarrow{m} & A \end{array}$$

and

$$\begin{array}{ccccc} I \otimes A & \xrightarrow{e \otimes 1} & A \otimes A & \xleftarrow{1 \otimes e} & A \otimes I \\ & {\scriptstyle \lambda_A} \searrow & \downarrow{\scriptstyle m} & \swarrow {\scriptstyle \rho_A} & \\ & & A & & \end{array}$$

commute. If $\otimes$ is symmetric we say that (A, m, e) is a *commutative* monoid if

$$\begin{array}{ccc} A \otimes A & \xrightarrow{\gamma_{A,A}} & A \otimes A \\ & {\scriptstyle m} \searrow \quad \swarrow {\scriptstyle m} & \\ & A & \end{array}$$

also commutes.

Examples 7.8.

(a) In $(\mathbf{Set}, \times, 1)$ monoids are just monoids in the usual sense. Similarly we can consider monoids in any category with finite products, e.g. **Top**. A monoid in **Cat** is a *strict* monoidal category.

(b) In a cocartesian monoidal category $(\mathcal{C}, \amalg, 0)$ every object has a *unique* (commutative) monoidal structure, given by the unique morphism $0 \to A$ and hte codiagonal map $(1_A, 1_A) : A \amalg A \to A$.

(c) In $(\mathbf{AbGp}, \otimes, \mathbb{Z})$ (commutative) monoids are (commutative) rings.

(d) In $[\mathcal{C}, \mathcal{C}]$ monoids are monads on $\mathcal{C}$.

(e) In Δ the object 1 has a monoid structure given by the unique maps $0 \to 1$ and $2 \to 1$. This is the "universal monoid": given any monoidal category $(\mathcal{C}, \otimes, I)$ the category of strong monoidal functors $\Delta \to \mathcal{C}$ is equivalent to the category of monoids in $\mathcal{C}$ by the functor sending $F : \Delta \to \mathcal{C}$ to $F(1)$. (Note that given a monoid (A, m, e) in $\mathcal{B}$ and a (lax) monoidal functor $F : \mathcal{B} \to \mathcal{C}$, FA

has a monoid structure given by $FA \otimes FA \xrightarrow{\theta} F(A \otimes A) \xrightarrow{Fm} FA$ and $I \xrightarrow{k} FI \xrightarrow{Fe} FA$.) Given a monoid (A, m, e) in $\mathcal{C}$ the morphisms

$$\underbrace{(\cdots(A \otimes A)\cdots) \otimes A}_{n \text{ factors}} \to \underbrace{(\cdots(A \otimes A)\cdots) \otimes A}_{m \text{ factors}}$$

obtainable by composing instances of m and e correspond to morphisms $n \to m$ in Δ.

11/29/06

There is also a universal commutative monoid, living in the category $\mathbf{Set}_f$ of finite sets and functors between them (with the cartesian monoidal structure): it is the terminal object $*$. Given a commutative monoid (A, m, e) in an arbitrary symmetric monoidal category $(\mathcal{C}, \otimes, I)$ the assignment $n \mapsto \underbrace{(\cdots(A \otimes A)\cdots) \otimes A}_{n \text{ factors}}$ can be made into a strong symmetric monoidal functor $\mathbf{Set}_f \to \mathcal{C}$.

Definition 7.9. Let $(\mathcal{C}, \otimes, I)$ be a monoidal category. We say the monoidal structure is *left closed* if, for each $A \in \operatorname{ob}\mathcal{C}$ $A \otimes \cdot : \mathcal{C} \to \mathcal{C}$ has a right adjoint. Similarly $\otimes$ is *right closed* if $\cdot \otimes A$ has a right adjoint. If both hold we say $\otimes$ is *biclosed*. For a symmetric monoidal structure $\otimes$ we simply say $\otimes$ is *closed* if it's left (equivalently right) closed. We write $[A, -]$ for the right adjoint of $\cdot \otimes A$. So we have natural bijections $\frac{A \to [B,C]}{A \otimes B \to C}$ (natural in A and C).

Examples 7.10.

(a) $(\mathbf{Set}, \times, 1)$ is closed. (We say $\mathcal{C}$ is *cartesian closed* if $(\mathcal{C}, \times, 1)$ is closed.) We know that functions $A \times B \to C$ correspond naturally to functions $A \to C^B$ (where C^B is the set of functions $B \to C$) so we set $[B, C] = C^B$.

(b) $\mathbf{Cat}$ is cartesian closed. Here we take $[\mathcal{C}, \mathcal{D}]$ to be the category of all functors $\mathcal{C} \to \mathcal{D}$ and it's easy to see that functors $\mathcal{B} \to [\mathcal{C}, \mathcal{D}]$ correspond to functors $\mathcal{B} \times \mathcal{C} \to \mathcal{D}$.

(c) For any small category $\mathcal{C}$ $[\mathcal{C}, \mathbf{Set}]$ is cartesian closed.

Proof 1. Use the Special Adjoint Functor Theorem: $\cdot \times F : [\mathcal{C}, \mathbf{Set}] \to [\mathcal{C}, \mathbf{Set}]$ preserves all small colimits, since limits and colimits are constructed pointwise. We know $[\mathcal{C}, \mathbf{Set}]$ is cocomplete and locally small, has a separating set $\{\mathcal{C}(A, -) \mid A \in \operatorname{ob}\mathcal{C}\}$ and it's well-copowered (since epimorphisms are pointwise surjective). □

Proof 2. Use the Yoneda Lemma. Whatever $[F, G]$ is, elements of $[F, G](A)$ must correspond to natural transformations $\mathcal{C}(A, \cdot) \to [F, G]$ and hence to natural transformations $\mathcal{C}(A, \cdot) \times F \to G$. So we define $[F, G](A) = [\mathcal{C}, \mathbf{Set}](\mathcal{C}(A, \cdot) \times F \to G)$. Given $f : A \to B$ we have $\mathcal{C}(f, \cdot) : \mathcal{C}(B, \cdot) \to \mathcal{C}(A, \cdot)$ and composition with $\mathcal{C}(f, \cdot) \times 1_r$ yields a mapping $[F, G](A) \to [F, G](B)$. This makes $[F, G]$ a functor. □

Exercise: verify that, for any H, natural transformations $H \to [F, G]$ corespond bijectively to natural transformations $H \times F \to G$.

(d) $(\mathbf{AbGp}, \otimes, \mathbb{Z})$ is closed: homomorphisms $A \otimes B \to C$ correspond to bilinear maps $A \times B \to C$ which in turn correspond to homomorphisms $A \to \mathbf{AbGp}(B, C)$ where $\mathbf{AbGp}(B, C)$ is equipped with the pointwise abelian group structure, i.e. $(f + g)(b) = f(b) + g(b)$. Similarly for $(\mathbf{Mod}_R, \otimes_R, R)$ if R is commutative, or more generally for any finitely generated abelian category $\mathcal{A}$ which is enriched over itself in "the obvious way".

(e) Let A be a fixed set and consider the poset $P(A \times A)$ of binary relations on A. Composition of relations defines a non-symmetric strict monoidal structure on $P(A \times A)$. This structure is biclosed: if we have a morphism $S \circ T \to R$ then $T \subseteq R/S$ where $R/S = \{(a, c) \mid \forall b \ (b, c) \in S \Rightarrow (a, b) \in R\}$. R/S is the largest relation such that $S \circ R/S \subseteq R$ i.e. $\cdot/S$ is right adjoint to $S \circ \cdot$.

Lemma 7.11. *In any closed monoidal category $\mathcal{C}$ the assignment $(B, C) \to [B, C]$ is a functor $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$ and the bijection $\frac{A \to [B,C]}{A \otimes B \to C}$ is natural in all three variables.*

Proof. Given $g : B' \to B$ and $h : C \to C'$ we define $[g, h] : [B, C] \to [B', C']$ to be the morphism corresponding to $[B, C] \otimes B' \xrightarrow{1 \otimes g} [B, C] \otimes B \xrightarrow{\mathrm{er}} C \xrightarrow{h} C'$ where er is the counit of $(\cdot \otimes B \dashv [B, \cdot])$. The rest is straightforward verification. □

We can now construct natural isomorphisms such as $[A, [B, C]] \cong [A \otimes B, C]$. We also have natural transformations $[B, C] \otimes [A, B] \to [A, C]$ corresponding to

$$[B, C] \otimes [A, B] \otimes A \xrightarrow{1 \otimes \mathrm{er}} [B, C] \otimes B \xrightarrow{\mathrm{er}} C$$

and $I \to [A, A]$ corresponding to $\lambda_A : I \otimes A \to A$. This defines an enrichment of $\mathcal{C}$ over itself, where we regard $\mathcal{C}(I, \cdot) : \mathcal{C} \to \mathbf{Set}$ as a "forgetful functor" sinces morphisms $I \to [A, B]$ correspond to morphisms $A \to B$.

8. Important things to remember

(i) The meaning of the Yoneda lemma.
(ii) What it means for (A, x) to be the representation of a functor. (Take the representation of $U : \mathbf{Gp} \to \mathbf{Set}$ as the usual example.)
(iii) Theorem 3.3 says that the naturality conditions in the definition of an adjunction mean that the image of A needs to be the limit of the morphisms leading out of it.
(iv) Special/General adjoint functor theorems.
(v) The domain and codomain of $\operatorname{im} f$ and $\operatorname{coim} f$ and what these actually mean.