$$
\frac{}{()\,:\,\mathbf{unit}}
\qquad
\frac{e_1:\alpha \quad e_2:\beta}{(e_1,e_2):\alpha\times\beta}
\qquad
\frac{e:\alpha\times\beta}{\mathbf{fst}\;e:\alpha}
\qquad
\frac{e:\alpha\times\beta}{\mathbf{snd}\;e:\beta}
\qquad
\frac{e:\alpha}{\mathbf{inl}\;e:\alpha+\beta}
\qquad
\frac{e:\alpha+\beta \quad \begin{array}{c} x:\alpha \\ \vdots \\ e':\mathbf{m}\,\gamma \end{array}}{\mathbf{let\;inl}\;x=e;\,e':\mathbf{m}\,\gamma}
\qquad
\frac{e:\beta}{\mathbf{inr}\;e:\alpha+\beta}
\qquad
\frac{e:\alpha+\beta \quad \begin{array}{c} x:\beta \\ \vdots \\ e':\mathbf{m}\,\gamma \end{array}}{\mathbf{let\;inr}\;x=e;\,e':\mathbf{m}\,\gamma}
$$

$$
\frac{r\in\mathbb{R}}{r:\mathbf{real}}
\qquad
\frac{e_1:\mathbf{real} \quad e_2:\mathbf{real}}{e_1+e_2:\mathbf{real}}
\qquad
\frac{e:\mathbf{real}}{-e:\mathbf{real}}
\qquad
\frac{e_1:\mathbf{real} \quad e_2:\mathbf{real}}{e_1\times e_2:\mathbf{real}}
\qquad
\frac{e:\mathbf{real}}{e^{-1}:\mathbf{real}}
\qquad
\frac{e:\mathbf{real}}{\exp e:\mathbf{real}}
\qquad
\frac{e:\mathbf{real}}{\log e:\mathbf{real}}
\qquad
\frac{e_1:\mathbf{real} \quad e_2:\mathbf{real}}{e_1<e_2:\mathbf{unit}+\mathbf{unit}}
$$

$$
\frac{e:\alpha}{\mathbf{dirac}\;e:\mathbf{m}\,\alpha}
\qquad
\frac{e:\mathbf{m}\,\alpha \quad \begin{array}{c} x:\alpha \\ \vdots \\ e':\mathbf{m}\,\beta \end{array}}{x\leftsquigarrow e;\,e':\mathbf{m}\,\beta}
\qquad
\frac{e:\mathbf{real} \quad e':\mathbf{m}\,\alpha}{e\rtimes e':\mathbf{m}\,\alpha}
\qquad
\frac{e_1:\mathbf{m}\,\alpha \quad e_2:\mathbf{m}\,\alpha}{e_1\oplus e_2:\mathbf{m}\,\alpha}
\qquad
\frac{}{\emptyset:\mathbf{m}\,\alpha}
\qquad
\frac{}{\mathbf{lebesgue}:\mathbf{m\;real}}
$$

Terms	e
Types	α,β,γ
Variables	x
Real numbers	$r,t \quad \in\mathbb{R}$
Head normal forms	$n ::= a \mid () \mid (e_1,e_2) \mid \mathbf{inl}\;e \mid \mathbf{inr}\;e \mid r \mid g[e] \mid \mathbf{dirac}\;e \mid e_1\oplus e_2 \mid \emptyset \mid \mathbf{lebesgue}$
Atomic terms	$a,b ::= x$ (not bound in the heap) $\mid \mathbf{fst}\;a \mid \mathbf{snd}\;a \mid a+n \mid n+a \mid -a \mid a\times n \mid n\times a \mid a^{-1} \mid \exp n \mid \log n \mid a<n \mid n<a$
Heaps	$h ::= [\,] \mid h[g[\,]]$
Bindings	$g ::= \mathbf{let\;inl}\;x=e;\,[\,] \mid \mathbf{let\;inr}\;x=e;\,[\,] \mid x\leftsquigarrow e;\,[\,] \mid e\rtimes[\,]$

Figure 1. Syntax and type system

Lazy partial evaluation to head normal form

$$
\begin{array}{lll}
\mathit{determine}\colon \mathit{Term}_{\mathbf{m}\alpha} \to (N_\alpha \to \mathit{Heap} \to N_{\mathbf{m}\gamma}) \to \mathit{Heap} \to N_{\mathbf{m}\gamma} & & \\
\mathit{determine}\ e\ c\ h & = x \leftsquigarrow e;\ (c\ x\ h) & \text{if } e \text{ is atomic or } \textbf{lebesgue} \\
\mathit{determine}\ (g[e])\ c\ h & = \mathit{determine}\ e\ c\ (h[g[\,]]) & \\
\mathit{determine}\ (e_1 \oplus e_2)\ c\ h & = (\mathit{determine}\ e_1\ c\ h) \oplus (\mathit{determine}\ e_2\ c\ h) & \\
\mathit{determine}\ \emptyset\ c\ h & = \emptyset & \\
\mathit{determine}\ (\mathbf{dirac}\ e)\ c\ h & = \mathit{evaluate}\ e\ c\ h & \\
\mathit{determine}\ e\ c\ h & = \mathit{evaluate}\ e\ (\lambda n.\, \mathit{determine}\ n\ c)\ h & \text{unless } e \text{ is in head normal form}
\end{array}
$$

$$
\begin{array}{lll}
\mathit{evaluate}\colon \mathit{Term}_{\alpha} \to (N_\alpha \to \mathit{Heap} \to N_{\mathbf{m}\gamma}) \to \mathit{Heap} \to N_{\mathbf{m}\gamma} & & \\
\mathit{evaluate}\ n\ c\ h & = c\ n\ h & \text{given } n \text{ already in head normal form} \\
\mathit{evaluate}\ (\mathbf{fst}\ e)\ c\ h & = \mathit{evaluate}\ e\ (\lambda n.\, c\ (\mathit{fst}\ n))\ h & \\
\mathit{evaluate}\ (\mathbf{snd}\ e)\ c\ h & = \mathit{evaluate}\ e\ (\lambda n.\, c\ (\mathit{snd}\ n))\ h & \\
\mathit{evaluate}\ (-e)\ c\ h & = \mathit{evaluate}\ e\ (\lambda n.\, c\ (\mathit{neg}\ n)) & \\
\mathit{evaluate}\ (e_1 + e_2)\ c\ h & = \mathit{evaluate}\ e_1\ (\lambda n_1.\, \mathit{evaluate}\ e_2\ (\lambda n_2.\, c\ (\mathit{add}\ n_1\ n_2)))\ h & \\
\mathit{evaluate}\ (\exp e)\ c\ h & = \mathit{evaluate}\ e\ (\lambda n.\, c\ (\exp n)) & \\
\mathit{evaluate}\ x\ c\ (h_1[x \leftsquigarrow e;\, h_2]) & = \mathit{determine}\ e\ (\lambda n.\, \lambda h_1'.\, c\ n\ (h_1'[x \leftsquigarrow \mathbf{dirac}\ n;\, h_2]))\ h_1 & \\
\mathit{evaluate}\ x\ c\ (h_1[\mathbf{let\ inl}\ x = e;\, h_2]) & = \mathit{evaluate}\ e\ (\lambda n'.\, \mathit{outl}\ n'\ (\lambda n.\, \lambda h_1'.\, c\ n\ (h_1'[x \leftsquigarrow \mathbf{dirac}\ n;\, h_2])))\ h_1 & \\
\mathit{evaluate}\ x\ c\ (h_1[\mathbf{let\ inr}\ x = e;\, h_2]) & = \mathit{evaluate}\ e\ (\lambda n'.\, \mathit{outr}\ n'\ (\lambda n.\, \lambda h_1'.\, c\ n\ (h_1'[x \leftsquigarrow \mathbf{dirac}\ n;\, h_2])))\ h_1 &
\end{array}
$$

Abstract operations on head normal forms

$$
\begin{array}{ll}
\mathit{fst}\colon N_{\alpha\times\beta} \to (N_\alpha \to \mathit{Heap} \to N_{\mathbf{m}\gamma}) \to \mathit{Heap} \to N_{\mathbf{m}\gamma} & \mathit{snd}\colon N_{\alpha\times\beta} \to (N_\beta \to \mathit{Heap} \to N_{\mathbf{m}\gamma}) \to \mathit{Heap} \to N_{\mathbf{m}\gamma} \\
\mathit{fst}\ a\ c = c\ (\mathbf{fst}\ a) & \mathit{snd}\ a\ c = c\ (\mathbf{snd}\ a) \\
\mathit{fst}\ (e_1, e_2)\ c = \mathit{evaluate}\ e_1\ c & \mathit{snd}\ (e_1, e_2)\ c = \mathit{evaluate}\ e_2\ c \\
 & \\
\mathit{neg}\colon N_{\mathbf{real}} \to N_{\mathbf{real}} & \mathit{add}\colon N_{\mathbf{real}} \to N_{\mathbf{real}} \to N_{\mathbf{real}} \\
\mathit{neg}\ r = \dot{-} r \in \mathbb{R} & \mathit{add}\ r_1\ r_2 = r_1 \dot{+} r_2 \in \mathbb{R} \\
\mathit{neg}\ a = -a & \mathit{add}\ n_1\ n_2 = n_1 + n_2 \quad \text{if } n_1 \text{ or } n_2 \text{ is atomic} \\
 & \\
\mathit{outl}\colon N_{\alpha+\beta} \to (N_\alpha \to \mathit{Heap} \to N_{\mathbf{m}\gamma}) \to \mathit{Heap} \to N_{\mathbf{m}\gamma} & \mathit{outr}\colon N_{\alpha+\beta} \to (N_\beta \to \mathit{Heap} \to N_{\mathbf{m}\gamma}) \to \mathit{Heap} \to N_{\mathbf{m}\gamma} \\
\mathit{outl}\ a\ c\ h = \mathbf{let\ inl}\ x = a;\ (c\ x\ h) & \mathit{outr}\ a\ c\ h = \mathbf{let\ inr}\ x = a;\ (c\ x\ h) \\
\mathit{outl}\ (\mathbf{inl}\ e)\ c\ h = \mathit{evaluate}\ e\ c\ h & \mathit{outr}\ (\mathbf{inr}\ e)\ c\ h = \mathit{evaluate}\ e\ c\ h \\
\mathit{outl}\ (\mathbf{inr}\ e)\ c\ h = \emptyset & \mathit{outr}\ (\mathbf{inl}\ e)\ c\ h = \emptyset
\end{array}
$$

Figure 2. Metalanguage functions (indicated by italic font) for lazy partial evaluation to head normal form