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Measure Theoretic Results for Continuous Valuations on Partially Ordered Spaces

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Abstract

The present work is an exploration in topological measure theory using domain theory as the main tool. Our main objects of study are continuous valuations on directed complete partially ordered (dcpo) sets. The term *valuation* designates a non-negative, extended real valued function ν, defined on the lattice of open sets of a topological space $(X, \mathcal{G})$ and satisfying: $\nu(\emptyset) = 0$ (strictness), $A \subseteq B$ implies $\nu(A) \subseteq \nu(B)$ (monotonicity) and $\nu(A \cup B) + \nu(A \cap B) = \nu(A) + \nu(B)$ (modularity). We say that ν is (Scott) *continuous* if for every increasing net of open sets $\{G_i\}_{i \in I}$ we have that $\nu(\bigcup_{i \in I} G_i) = sup_{i \in I} \nu(G_i)$.

We show the following results: if X is a monotone convergence space (e. g. a dcpo endowed with the Scott topology) and ν is the supremum of an increasing net of simple valuations, then ν can be extended uniquely to a τ-smooth Borel measure; for all T_0 non-sober spaces whose sobrification is first countable there exists a finite continuous valuation that does not extend to a measure; every locally finite continuous valuation on a regular Hausdorff space or a locally compact sober space (e. g. a continuous dcpo) extends uniquely to a Borel measure. In the latter case if the valuation is finite, then the extension is a perfect measure; this establishes the existence of disintegrations (regular conditional probability distributions) for stably locally compact spaces and factorizations for locally compact sober spaces under certain conditions.

We unify the two main approaches to the construction of integrals using domain theory, namely the Riemann-Edalat integral [Eda95b][EN98][How00] and the valuation integral [Jon89][Kir93][Tix95][Hec96] (for bounded functions and finite valuations). We characterise the probabilistic power domain $\mathbf{P}^1 X$ of a stably locally compact space X as a stochastically ordered space of probability measures, obtaining thus a generalisation of the Splitting Lemma [Jon89]. We show that if X is a stably compact space, then there exist topologies (coarser that the Scott topology) in the probabilistic powerdomains $\mathbf{P}^1 X$ and $\mathbf{P} X$ which make them stably compact spaces.

We study the problem of embedding a topological space $(X, \mathcal{G})$ into a pointed continuous dcpo $(D, \sqsubseteq)$ in such a way that the subspace Scott topology on X agrees with $\mathcal{G}$. We show that a net of τ-smooth probability measures on X converges weakly, if and only if, the induced net of continuous valuations in $\mathbf{P}^1 D$ converges in the Scott topology of $\mathbf{P}^1 D$. We show that for any locally compact sober space $(X, \mathcal{G})$ there exists a continuous dcpo $(\Lambda X, \sqsubseteq)$ such that the patch space $(X, \mathcal{G}_p)$ is homeomorphic to the set of maximal elements $\max(\Lambda X)$ with the relative Scott and Lawson topologies of ΛX.

Finally we give a characterisation of the maximal continuous functions from $\mathbf{I}\overline{\mathbb{R}}$ to $\mathbf{I}\overline{\mathbb{R}}$, where $\mathbf{I}\overline{\mathbb{R}}$ is the domain of compact extended-real intervals; we construct interval versions of the domain theoretic integrals and we give a result linking maximality in the function space and integrability in the sense of Riemann.

Keywords: coherent spaces; compact ordered spaces; continuous valuations; continuous domains; disintegration; domain representations; domain theoretic integration; extension of continuous valuations; formal lenses; generalised splittings; locally compact sober spaces; locally quasicompact sober spaces; maximal point spaces; monotone convergence spaces; patch topology; patch space; pospace; probabilistic powerdomains; probabilities with given marginals; regular conditional probability distributions; Riemann-Edalat integral; Scott topology; sober spaces; spaces of lenses; stably compact spaces; stochastic order; valuation integral; valuations; weak topology.

2000 Mathematics Subject Classification

06-XX **Order, lattices, ordered algebraic structures**

06Axx Ordered sets

06A06 Partial order, general

06Bxx Lattices

06B35 Continuous lattices and posets, applications

06Fxx Ordered structures

06F30 Topological lattices, order topologies

26-XX **Real functions**

26Axx Functions of one variable

26A42 Integrals of Riemann, Stieltjes and Lebesgue type

28-XX **Measure and integration**

28Axx Classical measure theory

28A12 Contents, measures, outer measures, capacities

28A25 Integration with respect to measures and other set functions

28A33 Spaces of measures, convergence of measures

28A50 Integration and disintegration of measures

28Bxx Set functions, measures and integrals with values in abstract spaces

28B15 Set functions, measures and integrals with values in ordered spaces

28Cxx Set functions and measures on spaces with additional structure

28C15 Set functions and measures on topological spaces (regularity of measures, etc.)

54-XX **General topology**

54Axx Generalities

54A10 Several topologies on one set

54Fxx Special properties

54F05 Linearly ordered topological spaces, generalized ordered spaces, and partially ordered spaces

54Hxx Connections with other structures, applications

54H12 Topological lattices, etc.

60-XX **Probability theory and stochastic processes**

60Bxx Probability theory on algebraic and topological structures

60B05 Probability measures on topological spaces

60B10 Convergence of probability measures

60Gxx Stochastic processes

60G99 None of the above, but in this section

This thesis is dedicated to my parents
Ethel and José Manuel

Summer Sunday Afternoon

It's an ordinary summer afternoon,
yet it's everything I ever wanted it to be.

Beyond the cotton wool clouds and the blue sky,
masses of cosmic matter and gas colliding.

But... Who cares?
the day is still the day and the night is still the night,
and we are here only for a little while.

Gunnersbury Park, London 1999

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The idea of relating probabilistic domains with stochastic orders sprang from joint work with Professor Edalat and Dr. Saheb-Djahromi ([AMESD00] and the material in Section 2.5). Professor Jimmie Lawson supplied the argument in the proof of Lemma 2.9. Dr. Philipp Sünderhauf pointed out inaccuracies in early versions of the proof of Lemma 2.30. Dr. Steve Vickers suggested that non-sober spaces would be a suitable place to look for counterexamples to the extension problem.

A big turning point in my research occurred during the visit to Darmstadt University of Technology, Germany in 1998. I want to acknowledge the hospitality of the University. Professor P. Burmeister let me his flat during my stay and indirectly contributed to the success of my work: when I finally understood the procedure for disposing of the rubbish the rest of the research appeared trivial. My research also benefited from discussions with Dr. Regina Tix.

Professor Ralph Kopperman provided Example 3.46. I also want to thank him for a series of engaging internet discussions on compactifications and the 'abnormal' definition of bitopological normality (Theorem 3.45). I would like to thank Dr. Mike Smyth for making me aware of his work on compactifications. Mr. Marek Sergot and Baljit **K**. made useful suggestions for improving the introduction of this thesis. Tom provided high level consultancy in English language grammar and usage. Prahlad, Reyes and Adriana lent me computer equipment at different stages of the writing up.

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Contents

Introduction

0.1 Motivation

The most succinct way of describing the present work is as an exploration in topological measure theory using domain theory as the main tool. Domain theory was introduced in the early 1970s by D. S. Scott [Sco70] in order to provide mathematical models for high-level programming languages. It allowed Scott to construct the first model for the untyped lambda calculus, a much celebrated result in mathematical logic and functional programming languages. Since then domain theory has become perhaps the most accepted tool for modelling the semantics of programming languages. Independently, Y. L. Ershov [Ers72] made important contributions to the area in the context of recursion theory. Another substantial part of the theory came from the theory of continuous lattices [GHK$^+$80]. Overall, domain theory can be thought of as a general theory of continuous partial orders where the governing notion is that of directed complete partial order (dcpo); that is, a partially ordered set where every directed subset has a least upper bound. The reader will find a quick general overview of domain theory in [Mis98]; more specialised references are provided in Section 1.3. Relatively recently however, with the seminal work of A. Edalat [Eda95d], domain theory acquired a different twist with important applications in pure and applied mathematics and physics. The scope of these applications ranges from iterated function systems and fractals [Eda96] to statistical physics [Eda95a] and stochastic processes [Eda95c] to mention some. Much of Edalat's work will be discussed as we proceed with our exposition; the interested reader can find a comprehensive overview in Edalat's own survey of the area [Eda97a].

The central notion in our work and main object of study is that of a continuous valuation. The term **valuation** is borrowed from Birkhoff [Bir67] who used it to designate a non-negative extended real valued function ν defined on a lattice $(0, 1, \vee, \wedge, \mathcal{L})$ and satisfying: $\nu(0) = 0$ (strictness), $A \leq B$ implies $\nu(A) \leq \nu(B)$ (monotonicity) and $\nu(A \vee B) + \nu(A \wedge B) = \nu(A) + \nu(B)$ (modularity). In the context of continuous lattices it became natural to require that $\nu(\bigvee^{\uparrow}_{j \in I} A_j) = \sup_{j \in I} \nu(A_j)$ whenever the directed join $\bigvee^{\uparrow}_{j \in I} A_j$ exists. In that case, the valuation is said to be (Scott) **continuous**. We will only consider continuous valuations defined on the lattice of open sets

$(\emptyset, X, \cup, \cap, \mathcal{G})$ of a topological space $(X, \mathcal{G})$. Thus, continuous valuations can be seen as a way of assigning a notion of content to the open sets in a topological space.

While the plain definition of valuation has only limited power, when enhanced with continuity it becomes a rich mathematical object with many interesting properties. It is the author's belief that continuous valuations are a more suitable paradigm for studying the notion of content in a topological space than measures[1]. The reason is that neither the definition of topology nor the definition of valuation has any explicit reference to cardinality. On the other hand the concepts of σ-algebra and measure are hopelessly stitched to the cardinality of the natural numbers. As an example supporting our thesis we can mention Heckmann's purely topological version of the Riesz Representation Theorem (part (7) of Proposition 6.20 below). Thus a considerable amount of the work presented here focuses on determining the extent to which the results that are known for measures hold for continuous valuations and vice versa (e. g. Chapters 4 and 7). The topological spaces used in domain theory, and hence many of those appearing in computer science, are typically non-Hausdorff. So it is our feeling that to a large extent classical probability and measure theory has an analogous formulation in continuous dcpo's and locally compact sober spaces. Thus, another important part of our work has to do with converting classical results for Hausdorff spaces to a non-Hausdorff setting.

Perhaps the most important application of continuous valuations is that they can be used to define integrals with strong continuity properties. We encase these recent developments under the name "domain theoretic integration". The Riemann-Edalat (R-) integral was introduced by Edalat in [Eda95b] as a generalisation of the Riemann integral for compact metric spaces. Later it was extended by Edalat and Negri [EN98] to locally compact Hausdorff spaces, and recently by Howroyd [How00] to any Hausdorff space $(X, \mathcal{G})$ that can be embedded as a subspace of a pointed continuous dcpo. The main feature of this approach is that the value of the integral can be approximated by upper and lower Riemann style sums. We use the name 'valuation integral' to refer to the work of Jones and Plotkin [Jon89][JP89], Kirch [Kir93], Tix [Tix95], and Heckmann [Hec96]. The valuation integral is defined for a Scott continuous function with respect to a continuous valuation on any topological space. The way in which continuous valuations are used to con-

[1] In this regard Topsøe [Top70b, p. 74] had already pointed out that "all abstract τ-smooth measure theory is topological."

struct these integrals is different, but in both cases it becomes crucial to determine whether a continuous valuation can be extended uniquely to a Borel measure, in which case the integrals agree with the Lebesgue integral. This brings us to the following important problem which we study in Chapters 2 and 3.

Extension problem: Given a topological space $(X, \mathcal{G})$ and a continuous valuation ν on X. Consider the Borel σ-algebra $\sigma(\mathcal{G})$. Is it possible to extend ν uniquely to a Borel measure $\overline{\nu}$ on $(X, \sigma(\mathcal{G}))$?

For non-continuous valuations (according to Z. Lipecki) this problem was studied by F. Hausdorff [Hau49] back in 1914. The extension problem was also considered by Smiley [Smi44] for valuations on metric distributive lattices. Horn and Tarski [HT48] obtained other solutions as they studied measures in Boolean algebras. Pettis [Pet51] was probably the first author to formulate the extension problem for abstract sets; he also gave sufficient conditions for the extension to hold on Hausdorff spaces. Norberg [Nor89] established the results for σ-finite valuations on second countable locally compact sober spaces; Lawson [Law82] for finite valuations on these same spaces and stably compact spaces, which implies the extension for compact ordered topological spaces as defined in Nachbin [Nac65]. Norberg and Vervaat [NV97] gave an extension theorem for capacities which implies the extension result for locally finite valuations on stably compact spaces.

From the computer science point of view the problem reappeared in the context of probabilistic non-determinism in the work of Saheb-Djahromi [SD80] who gave a proof for ω -algebraic dcpo's. Unfortunately the proof contained a gap, which was propagated through the subsequent work of Jones and Plotkin ([JP89, Theorem 8.2] and [Jon89, Section 5.3]) for continuous dcpo's. This fact was pointed out by R. Tix and O. Kirch. The second chapter of this thesis comprises joint work with Professor A. Edalat and Dr. Saheb-Djahromi [AMESD00] which corrects the gap in [SD80] and gives a general result which implies the extension result for continuous dcpo's.

Many authors have studied the extension problem from different areas of mathematics. There are a good number of reasons why the theory of partial orders is a good approach. Let us start by discussing a simple example which will be used later to illustrate other facts. Every T_0 topological space carries an inherent partial order $\sqsubseteq_s$ known as the specialisation order. This order is given by $x \sqsubseteq_s y$ if x belongs to the closure of $\{y\}$. For T_1 spaces this order is the identity.

Example 0.1 *Consider the set of natural numbers* $(\mathbb{N}, \leq)$ *as a partially ordered set endowed with the Scott topology*[2] $\mathcal{G}_\sigma$, *i.e.* $\mathcal{G}_\sigma = \{\uparrow n \mid n \in \mathbb{N}\}$ *where* $\uparrow n = \{m \mid n \leq m\}$. *Clearly the specialisation order coincides with* $\leq$. *Define the valuation* ν *as* $\nu(G) = 1$ *for all* $G \neq \emptyset$ *and* $\nu(\emptyset) = 0$. *Then* ν *is a finite continuous valuation that does not extend to a measure since* $\bigcap_n \uparrow n = \emptyset$ *but* $\lim_n \nu(\uparrow n) = 1$. *On the other hand if we consider* $\mathbb{N} \cup \{\infty\}$ *with the Scott topology, then* ν *extends uniquely to the Dirac measure* δ_∞.

Hence, we are not interested in topological spaces $(X, \mathcal{G})$ where $(X, \sqsubseteq_s)$ is not a dcpo. Unfortunately even if we require $(X, \sqsubseteq_s)$ to be dcpo, this is still not enough since we can construct an example of a continuous valuation on a dcpo that does not extend uniquely to a measure (see Section 2.7). Nevertheless this dcpo with its Scott topology is not a sober space. The third chapter in this thesis gives a positive solution for the extension problem on locally compact sober spaces.

Another part of our work (Chapter 5) explores domain representations for topological spaces. For our purposes a domain representation will be a subspace embedding from a topological space into a continuous dcpo topologised with the Scott topology. We refer to such structures as S-domain representations to distinguish them from other existing notions of domain representations [SHT95][SSH97][Bla97b][Bla97a]. We are specially interested in the case where a topological space can be realised as a subset of the maximal elements of a continuous dcpo [WS81][EH98][Law97][Law98][Mar98][Mar00]. The interest in these kinds of representations arises from Edalat's work in integration [Eda95d][Eda95b] and the recent extensions of Howroyd [How00] to the theory.

In the final part of our work we turn our attention to functions taking values in $\mathbf{I}\overline{\mathbb{R}}$, the domain of compact extended-real intervals. We construct interval versions of the domain theoretic integrals and analyse the problem of maximality in the space of continuous functions and its relation with integration.

0.2 Overview

The codes for the 2000 Mathematics Subject Classification can be found in page 3.

Chapter 1: Notation and preliminary material.

[2] In this case, the Scott topology coincides with the upper Alexandroff topology.

Chapter 2: We show that if a locally finite continuous valuation on a monotone convergence space is the supremum of a directed family of simple valuations, then it extends uniquely to a τ-smooth Borel measure (note that any dcpo endowed with the Scott topology is a monotone convergence space). As an important corollary, every locally finite continuous valuation on a continuous dcpo extends uniquely to a Borel measure. We construct a finite continuous valuation on a dcpo (endowed with the Scott topology) that does not extend to a Borel measure. In fact, we show that this is the case for all T_0 and non-sober spaces whose sobrification is first countable.

2000 Mathematics Subject Classification: 06A06, 06B35, 06F30, 28B15, 28C15, 54F05, 60B05.

Chapter 3: Suppose that X is a regular Hausdorff (T_3) space or a locally compact sober space. We show that every locally finite continuous valuation on X extends uniquely to a Borel measure. As corollaries we obtain the extension result for metric spaces (known) and for continuous dcpo's (again). We discuss some bitopological properties of locally compact sober spaces.

2000 Mathematics Subject Classification: 06A06, 06B35, 06F30, 28A12, 28C15, 54A10, 54F05, 54H12, 60B05.

Chapter 4: Suppose that X is a locally compact sober topological space. We show that the extension of a finite continuous valuation on X to a Borel measure is a perfect measure (in particular, this holds for finite continuous valuations on continuous dcpo's). Since on stably locally compact spaces the extension of a finite valuation to a measure is compact, we can assert the existence of disintegrations (regular conditional probability distributions) and factorizations under certain conditions.

2000 Mathematics Subject Classification: 06A06, 06B35, 28A25, 28A50, 28C15, 60B05.

Chapter 5: We study the problem of embedding a topological space $(X, \mathcal{G})$ as a subspace of a continuous dcpo $(D, \sqsubseteq)$ in such a way that the subspace Scott topology on X agrees with $\mathcal{G}$. We present two constructions. The first one embeds any locally compact sober space $(X, \mathcal{G})$ as a subspace of a continuous lattice $(A(X), \sqsubseteq)$. The second one shows that for any locally compact sober space $(X, \mathcal{G})$ there exists a continuous dcpo $(\Lambda X, \sqsubseteq)$ such that the patch space $(X, \mathcal{G}_p)$ is homeomorphic to $\max(\Lambda X)$ with the relative Scott and Lawson topologies of ΛX. Furthermore, if X is stably locally compact, then ΛX is a coherent domain; if X is second countable, then ΛX is ω-continuous; and if X is stably compact, then ΛX has a bottom element. If

X is stably locally compact, then the space of lenses $(Lens(X), \supseteq)$ is also a coherent domain with the same properties as ΛX. If X is locally compact and Hausdorff, then $(Lens(X), \supseteq)$ agrees with the upper space of X.

2000 Mathematics Subject Classification: 06A06, 06B35, 06F30, 28C15, 54A10, 54F05, 54H12.

Chapter 6: We unify the two main approaches to the construction of integrals using domain theory, namely the Riemann-Edalat integral [Eda95b][EN98][How00] and the valuation integral [Jon89][Kir93][Tix95][Hec96]. Restricting ourselves to bounded real-valued functions and finite valuations we construct upper and real-valued versions of the valuation integral. Suppose that $(X, \mathcal{G})$ is a Hausdorff topological space which embeds as a subspace of a pointed continuous dcpo $(D, \sqsubseteq)$ in such a way that the subspace Scott topology on X agrees with $\mathcal{G}$. Let μ be a τ-smooth (Borel) probability measure on X and μ_e the continuous valuation on D induced by the embedding. We show that the upper and lower semi-continuous envelopes f_E and f^E of any bounded real-valued function f on X can be extended to lower and upper semi-continuous functions f_D and f^D whose domain is D. The lower and upper valuation integrals of f_D and f^D with respect to μ_e agree with the lower and upper R-integrals of f with respect to μ. Moreover, they also agree with the Lebesgue integrals $\mu(f_E)$, $\overline{\mu}_e(f_D)$ and $\mu(f^E)$, $\overline{\mu}_e(f^D)$ where $\overline{\mu}_e$ is the (unique) Borel measure which extends μ_e.

2000 Mathematics Subject Classification: 06A06, 06B35, 06F30, 26A42, 28A12, 28A25, 28B15, 28C15, 54F05, 60B05

Chapter 7: Under the same assumptions as in Chapter 6, we show that a net $(\overline{\mu}^j)_{j \in J}$ of τ-smooth probability measures on X converges in the weak topology to a τ-smooth probability measure $\overline{\mu}$ if and only if the net $(\mu_e^j)_{j \in J}$ converges to μ_e in the Scott topology of $\mathbf{P}^1 D$. Suppose that X is a stably locally compact space. Let ν and μ be normalised continuous valuations on X, and let $\overline{\nu}$ and $\overline{\mu}$ denote its unique extensions to Radon probability measures. We show that $\nu \sqsubseteq \mu$ if and only if $\overline{\nu}$ is stochastically smaller than $\overline{\mu}$, in which case there exists a continuous valuation $\lambda \in \mathbf{P}^1(X \times X)$ with $\pi_1[\lambda] = \nu$, $\pi_2[\lambda] = \mu$ and $\overline{\lambda}(\{(x,y) \mid x \sqsubseteq_s y\}) = 1$ (Generalised Splitting Lemma). Finally we show that if X is a stably compact space, then there exist topologies (coarser that the Scott topology) in the probabilistic powerdomains $\mathbf{P}^1 X$ and $\mathbf{P} X$ which make them stably compact spaces. This solves an open question raised by Jung and Tix in [JT98].

2000 Mathematics Subject Classification: 06A06, 06B35, 06F30, 26A42, 28A12, 28A33, 28C15, 54F05, 54H12, 60B05, 60B10, 60G99.

Chapter 8: We study functions taking values in $\mathbf{I}\overline{\mathbb{R}}$, the domain of compact extended-real intervals. We construct interval valued versions of the integrals discussed in Chapter 6 and show that the corresponding unification result still holds. We exhibit a large family of functions from at T_0 topological space Y to a complete continuous semilattice L which are maximal in the space of continuous functions $[Y \to L]$. In particular, this allows us to characterize the maximal continuous functions in $[\mathbf{I}\overline{\mathbb{R}} \to \mathbf{I}\overline{\mathbb{R}}]$. We say that a function $f : \mathbb{R} \to \mathbb{R}$ is maximal representable if there exists a maximal continuous function $F : \mathbf{I}\mathbb{R} \to \mathbf{I}\mathbb{R}$ such that $[f(x), f(x)] = F([x, x])$ for all $x \in \mathbb{R}$. We show that Riemann integrable functions are maximal representable but the converse is not true.

2000 Mathematics Subject Classification: 06B35, 26A42, 28B15, 54F05.

0.3 Organisation and style

Overall our intention has been to present the material in the most accessible way for the busy researcher who might only be interested in specific areas of our work. We have tried to keep the material in the thesis as self contained as possible. In Chapter 1 we have placed the preliminary material which is used throughout the thesis; many basic definitions and results have been included for reference and comparison with pointers to the relevant literature in the area. Nevertheless, we have never intended to define "everything". Hence, some familiarity with the techniques in domain theory, topology and measure theory will become inevitable. One of the things the author noticed when surveying different areas of mathematics is how relative the word "obvious" can be. Thus, the domain theoreticians will have to bear the boring references to the interpolation property, while the analysts will have to skip over the basic properties of function envelopes.

Accordingly we have tried to keep each chapter as self contained as possible. Each chapter begins with a summary section where the main results are stated and some background to the problem is given. Next the preliminary material particular to that chapter is presented. Sections not containing new results (or containing only minor ones) have been labelled "A review of..." At the risk of becoming redundant, we have placed constant reminders about the notation and definitions in each section. In this regard the spirit of our presentation runs counter to that of König [Kön97] for instance. We have also avoided as much as possible the use of mathematical notation in the statement of the main results, and stated the full set of assumptions for the result (as opposed to guessing them from the contextual information).

The reader will find a notation index and an alphabetical index towards the end. Every definition in the thesis has an index entry. A list of problems appears alongside the text. We note that their level of difficulty could vary considerably; some of them might have a straightforward solution while others state unresolved questions that the author could not solve after repeated attempts. This fulfills the purpose of giving further directions for our work.

1. Preliminaries

1.1 Special conventions

The following notational conventions are important; the objects in question are defined in the following sections. Unless otherwise specified, all topological spaces are assumed to be T_0 and all the order theoretical properties of a topological space are with respect to the specialisation order $\sqsubseteq_s$. If $(X, \mathcal{G})$ is a topological space, then $\mathcal{G}X$, $\mathcal{F}X$, $\mathcal{K}X$, $\mathcal{Q}X$ are the pavings of open, closed, compact and compact saturated subsets of X. When there is no confusion we write $\mathcal{G}$, $\mathcal{F}$, $\mathcal{K}$, $\mathcal{Q}$. If not specified otherwise, the sets denoted by the letters G, F, K, Q are assumed to be elements of the pavings $\mathcal{G}$, $\mathcal{F}$, $\mathcal{K}$, $\mathcal{Q}$ respectively. The cocompact, patch, Scott, upper, lower, Lawson and Alexandroff topologies on $(X, \mathcal{G})$ are denoted as $\mathcal{G}_{cc}$, $\mathcal{G}_p$, $\mathcal{G}_\sigma$, $\mathcal{G}_u$, $\mathcal{G}_l$, $\mathcal{G}_\lambda, \mathcal{G}_A$. The indices extend to: (i) the pavings $\mathcal{F}$, $\mathcal{K}$, $\mathcal{Q}$; (ii) the set functions $int(\_)$, $cl(\_)$ and $sat(\_)$; (iii) the Borel σ-algebras generated by the topologies. For example: $\mathcal{F}_p$ is the paving of patch closed sets; $cl_\lambda(E)$ is the closure of E with respect to the Lawson topology; $\mathcal{B}_{cc}X$ is the Borel σ-algebra $\sigma(\mathcal{G}_{cc}X)$. The corresponding superscripts also indicate which topology are we considering in the space; for example, $[X^p \to X^{cc}]$ is the set of continuous functions from $(X, \mathcal{G}_p X)$ to $(X, \mathcal{G}_{cc} X)$.

Sets are often used as subindices to the symbols $\sqsubseteq$, $\ll$, $\uparrow$, $\downarrow$ and $\Uparrow$ as to distinguish their scope. In particular, the letters $\mathcal{G}$, $\mathcal{F}$, $\mathcal{K}$, $\mathcal{Q}$ refer implicitly to properties of the posets $(\mathcal{G}, \subseteq)$, $(\mathcal{F}, \supseteq)$, $(\mathcal{K}, \supseteq)$ and $(\mathcal{Q}, \supseteq)$. For example, let $(P, \sqsubseteq)$ be a poset and $F \in \mathcal{F}_\sigma P$. That is, F is a Scott closed subset of P. Then

$$\begin{aligned} \downarrow F &= \{y \mid \exists x.\ x \in F \text{ and } y \sqsubseteq x\} \\ &= F \end{aligned}$$

since Scott closed sets are lower. While

$$\begin{aligned} \downarrow_{\mathcal{F}} F &= \downarrow_{\mathcal{F}} \{F\} \\ &= \{A \in \mathcal{F}_\sigma \mid A \sqsubseteq_{\mathcal{F}} F\} \\ &= \{A \in \mathcal{F}_\sigma \mid A \supseteq F\}. \end{aligned}$$

1.2 Basic notation and set theory

We use the notation $A := B$ to indicate that A is defined as B. Similarly, we write $A \equiv B$ to indicate that $A = B$ by definition. $\mathbb{R}$ denotes the set of real numbers, $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$ is the set of extended real numbers, $\mathbb{R}_+ := [0, \infty)$ and $\overline{\mathbb{R}}_+ := [0, \infty]$. In a similar way $\mathbb{N}$ denotes the set of natural numbers and $\overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$, $\mathbb{N}_+ := \mathbb{N} \backslash \{0\}$, $\overline{\mathbb{N}}_+ := \mathbb{N}_+ \cup \{\infty\}$. Notice that $0 \in \mathbb{R}_+$ but $0 \notin \mathbb{N}_+$. The sets of integers and rational numbers are denoted as $\mathbb{Z}$ and $\mathbb{Q}$ respectively. Let Ω be any set, then $2^\Omega := \{A \mid A \subseteq \Omega\}$ is the **power set** of Ω. We write $F \subseteq_f \Omega$ if F is a finite subset of Ω and $2^\Omega_f := \{A \mid A \subseteq_f \Omega\}$. Any $\mathcal{P} \subseteq 2^\Omega$ will be called a **paving** on Ω. Let $\mathcal{P}$ be a paving on Ω and X be any subset of Ω, we say that a paving $\mathcal{C}$ is a $\mathcal{P}$-**covering** for X if $\mathcal{C} \subseteq \mathcal{P}$ and $X \subseteq \bigcup \mathcal{C}$. The **identity function** on Ω is denoted by Id_Ω or simply by Id when there is no confusion. Let $f : A \to B$ be any function. If $C \subseteq A$, then $f[C] = \{f(x) \mid x \in C\}$ (also denoted as fC when there is no confusion) is the **image** of C under f, and $f \restriction_C : C \to B$ is the **restriction** of f to C. Similarly, if $D \subseteq B$ then $f^{-1}D := \{x \in A \mid f(x) \in D\}$ is the **inverse image** of D under f. Let $\mathcal{P}$ be a paving on Ω and Ω_0 be any subset of Ω we define the **trace** of $\mathcal{P}$ on Ω_0 as $tr(\mathcal{P} \mid \Omega_0) := \{E \cap \Omega_0 \mid E \in \mathcal{P}\}$; we also write $\Omega_0 \cap \mathcal{P}$. If (E_n) is a sequence of subsets of Ω then $\limsup_n E_n := \bigcap_{m \in \mathbb{N}} \bigcup_{n \geq m} E_n$ and $\liminf_n E_n := \bigcup_{m \in \mathbb{N}} \bigcap_{n \geq m} E_n$. That is, $\limsup_n E_n = \{x \mid x \in E_n$ for infinitely many $n \in \mathbb{N}\}$ and $\liminf_n E_n = \{x \mid x \in E_n$ for all but finitely many $n \in \mathbb{N}\}$.

Let $\{\Omega_j\}_{j \in I}$ be a family of sets indexed by a set I. Then each element $(\omega_j)_{j \in I}$ of the Cartesian product $\Pi_j \Omega_j$ is by definition a function $\omega : I \to \bigcup_{j \in I} \Omega_j$ where $\omega_j \in \Omega_j$; here ω_j is simply a shorthand for $\omega(j)$. For $A \subseteq B \subseteq I$ the natural **projection** $\pi_{BA} : \Pi_{j \in B} \Omega_j \to \Pi_{j \in A} \Omega_j$ is defined as $\pi_{BA}((\omega_j)_{j \in B}) := (\omega_j)_{j \in A}$. The projection π_{IA} is abbreviated to π_A and $\pi_{\{j\}}$ is abbreviated further to π_j. We use the following standard notation in measure theory as defined in [Top78] and [HJ94], where the symbols

$$\cup, \sum, \cup^{\uparrow}, \cap, \cap_{\downarrow}, \mathsf{c}, \backslash$$

stand for **union, disjoint union, directed union, intersection, filtered intersection, complementation**, and **set difference.** The modifiers: f, c stand for finite and countable. For example, a $(\emptyset, \Omega, \cup, \cap f)$-paving on Ω is a paving having $\emptyset$, Ω as elements and closed under arbitrary unions and finite intersections; i.e., a topology on Ω. In the same way a $(\emptyset, C, \cup\mathsf{c})$-paving on Ω is a σ-algebra on Ω. We also say, for example, that a topology is an

$(\cap f)$-**stable** paving. We make the convention that $(\cap f)$ is a finite nonempty intersection.

A paving $\mathcal{P}$ will be called **directed** if it is directed with respect to $\subseteq$, i.e. for all A_1, A_2 in $\mathcal{P}$ there exists A_3 in $\mathcal{P}$ such that $A_1 \cup A_2 \subseteq A_3$. The union of a directed paving is denoted by $\bigcup^{\uparrow} \mathcal{P}$. If $Y = \bigcup^{\uparrow} \mathcal{P}$, we also write $\mathcal{P}{\uparrow}Y$. In the same way, a paving is said to be **filtered** if it is filtered with respect to $\subseteq$; we use $\bigcap_{\downarrow} \mathcal{P}$ or $\mathcal{P}{\downarrow}Y$ to symbolize the filtered intersection of $\mathcal{P}$. Notice that for any paving $\mathcal{P}$ we have that $\bigcup \mathcal{P} = \bigcup^{\uparrow}\{\bigcup_{i=1}^{n} x_i \mid n \in \mathbb{N}$ and $x_i \in \mathcal{P}$ for $i = 1,..., n\}$. It follows that on $(\cup f)$-stable pavings the unions can always be assumed to be directed, and on $(\cap f)$-stable pavings the intersections filtered.

A **tree** on a set Ω is a set $T \subseteq \bigcup_{n \in \mathbb{N}_+} \Omega^n$ such that if $(\omega_1,..., \omega_k) \in T$ then $(\omega_1,..., \omega_j) \in T$ for all $j \leq k$. That is, T is a subset of the finite sequences in Ω such that it contains all initial segments of each of its elements. Elements of T are called nodes. A sequence (ω_n) in Ω is an **infinite branch** (or path) of T if $(\omega_1,..., \omega_k) \in T$ for all $k \in \mathbb{N}$ (notice that (ω_n) is not an element of T). A tree T is **finitely branching** if $\{(\omega_1) \in T\}$ is finite and for every $(\omega_1,..., \omega_k) \in T$ the set $\{(\omega_1,..., \omega_k, y) \in T \mid y \in \Omega\}$ is finite.

Proposition 1.1 *(König's Lemma) Let T be a finitely branching tree on a set Ω. Then T is infinite if and only if it has an infinite branch.*

Proof See [Mos80, 4F.9] for example. ■

König's Lemma is one of the many countable versions of the Axiom of Choice; other equivalences can be found in Chapter 0 of [Kel55], Section 7F of [Mos80, 4F.9], and Sections 4.20 and 9.11 of [DP90].

1.3 Partial orders and domain theory

Our main references for domain theory are [AJ94][Joh82][GHK+80]. The non-initiated reader can find a comprehensive introduction to the subject in [DP90][Law88][Mis98][Eda97a]. A **preorder** is a pair $(P, \sqsubseteq)$, where P is a set and $\sqsubseteq$ is a binary relation on P satisfying: $a \sqsubseteq a$ (reflexivity); $a \sqsubseteq b$ and $b \sqsubseteq c$ implies $a \sqsubseteq c$ (transitivity) for all $a, b, c \in P$. We also say that P is preordered by $\sqsubseteq$. The **opposite** preorder $(P, \sqsubseteq^{op})$ is defined as $a \sqsubseteq^{op} b$ if and only if $b \sqsubseteq a$. A preorder $(P, \sqsubseteq)$ is called a **partially ordered set** (poset) if we also have: $a \sqsubseteq b$ and $b \sqsubseteq a$ implies $a = b$ (antisymmetry). Given any preorder $(P, \sqsubseteq)$ we can obtain a poset in a canonical way by taking the quotient space $P/\sim$, where $a \sim b$ if and only if $a \sqsubseteq b$ and $b \sqsubseteq a$. Let $(P, \sqsubseteq)$ be a preorder. We write $R_{\sqsubseteq}$ when we want to refer to a property of the binary relation $\sqsubseteq$ as a subset of $P \times P$, i.e. $R_{\sqsubseteq} := \{(x, y) \mid x \sqsubseteq y\}$. For $A \subseteq P$ we define $\downarrow A := \{x \in P \mid \exists a \in A.x \sqsubseteq a\}$. We abbreviate $\downarrow\{a\}$ to $\downarrow a$. In a similar way we let $\uparrow A := \{x \in P \mid \exists a \in A.a \sqsubseteq x\}$ where $\uparrow\{a\}$ abbreviates to $\uparrow a$. A set $A \subseteq P$ is **lower** if $A = \downarrow A$ and **upper** if $A = \uparrow A$. An element $a \in P$ is **maximal** if $\{a\} = \uparrow a$ and **minimal** if $\{a\} = \downarrow a$. $\max(P)$ and $\min(P)$ denote the (possibly empty) subsets of maximal and minimal elements of P. Let F be a nonempty subset of P. F is said to be **directed** if for all $x, y \in F$ there exists $z \in F$ such that $x, y \sqsubseteq z$. F is said to be **filtered** if for all $x, y \in F$ there exists $z \in F$ such that $z \sqsubseteq x, y$. F is a **chain** or **totally ordered** set if for all $x, y \in F$ either $x \sqsubseteq y$ or $y \sqsubseteq x$. Let $(P, \sqsubseteq)$ be a poset. If P has a least element, that is an element which is below every element of P, then it is denoted by $\perp$ or 0 and P is said to be a **pointed** poset. In the same way the greatest element (when it exists) is denoted by $\top$ or 1. A poset P is **bounded complete** if each bounded pair $x_1, x_2 \in P$ has a least upper bound $x_1 \vee x_2$. A poset P is a **directed complete partial order** (dcpo) if every directed subset $D \subseteq P$ has a least upper bound (lub or supremum) denoted by $\bigsqcup^{\uparrow} D$. The following characterisation depends on the Axiom of Choice.

Proposition 1.2 *A poset $(P, \sqsubseteq)$ is a directed complete partial order if and only if every chain in P has a least upper bound.*

Proof See Corollary 1 in [Mar76]. ■

A poset is called a **lattice** if every nonempty finite subset F has a lub $\bigvee F$ and a greatest lower bound (glb or infimum) $\bigwedge F$. Notice that $\top = \bigwedge \emptyset$ and $\perp = \bigvee \emptyset$ when they exist. A lattice such that $\bigvee F$ and $\bigwedge F$ exist for all $F \subseteq P$ is called a **complete lattice**. Suppose that $(P, \sqsubseteq)$ is a poset and $x, y \in P$. We say that x approximates y or x is **way below** y, denoted by $x \ll y$, if whenever $A \subseteq P$ is a directed subset, $\bigsqcup^{\uparrow} A$ exists and $y \sqsubseteq \bigsqcup^{\uparrow} A$, then there exists $a \in A$ with $x \sqsubseteq a$. If $C \subseteq P$ we define $\twoheaduparrow C := \{x \in D \mid \exists c \in C.c \ll x\}$. We abbreviate $\twoheaduparrow\{c\}$ to $\twoheaduparrow c$. In a similar way we define $\twoheaddownarrow C$. We say $B \subseteq P$ is a **basis** if for every $x \in P$, the set $B_x := \{b \in B \mid b \ll x\}$ is directed and $x = \bigsqcup^{\uparrow} B_x$. If P has a basis it is called a **continuous poset** [Mis98, Definition 4.15]; or **ω-continuous poset** if the basis is countable. A dcpo $(D, \sqsubseteq)$ which is a continuous poset is called a **continuous dcpo**. Following Abramsky and Jung [AJ94] we use the word **domain** as a synonym for continuous dcpo. The reader should be warned that the terminology in the area is not standard: continuous dcpo's often appear in the literature as continuous posets, while some authors [Mis98][SHLG94] reserve the term domain for more restrictive settings. A complete lattice which is continuous is called a **continuous lattice**. A continuous dcpo $(D, \sqsubseteq)$ with least element is called an **L-domain** (lattice domain) if each pair $x_1, x_2 \in D$ bounded by $x \in D$ has a least upper bound in $\downarrow x$; this is equivalent to saying that the set $\downarrow x$ is a continuous lattice for all $x \in D$. A bounded complete continuous dcpo is called **bc-domain**; every pointed bc-domain is clearly an L-domain. The following is probably the most important property of continuous dcpo's.

Proposition 1.3 *(Interpolation property) Let $(D, \sqsubseteq)$ be a continuous dcpo and $M \subseteq_f D$ be a finite subset of D. If $y \in D$ and $x \ll y$ for all $x \in M$, then there exists $z \in D$ such that $x \ll z \ll y$ for all $x \in M$. Furthermore, if B is a basis for D, then z might be chosen from B. Conversely if there exists a set B' such that when $x \ll y$ there exists $z \in B'$ with $x \ll z \ll y$, then B' is a basis for D.*

Proof See Lemma 2.2.15 in [AJ94]. ■

Let $(D_j, \sqsubseteq_j)_{j \in I}$ be a family of dcpo's. For $(x_j)_{j \in I}$, $(y_j)_{j \in I}$ in $\Pi_j D_j$ let the **Cartesian product** order be defined as $(x_j)_{j \in I} \sqsubseteq (y_j)_{j \in I}$ if $x_j \sqsubseteq_j y_j$ for all $j \in I$. Then $(\Pi_j D_j, \sqsubseteq)$ is a dcpo where $\bigsqcup^{\uparrow}\{(x_j^{\alpha})_{j \in I} \mid \alpha \in J\} = (\bigsqcup^{\uparrow}\{x_j^{\alpha} \mid \alpha \in J\})_{j \in I}$.

1.4 General topology

Our main references are [Dug66][Bou66]. Let X be any set. A $(\emptyset, X, \cap f, \cup)$-paving $\mathcal{G}$ on X is called a **topology** on X. A topological space is a pair $(X, \mathcal{G})$ where $\mathcal{G}$ is a topology on X. For the rest of the section $(X, \mathcal{G})$ denotes a topological space. If $(Y, \mathcal{H})$ is another topological space, then the **set of continuous functions** from X to Y is denoted as $[X \to Y]$. $\mathcal{F}$ denotes the paving of **closed** subsets of X. For any $A \subseteq X$ the **interior** and the **closure** of A are denoted as $int(A)$ and $cl(A)$ respectively. For each $y \in X$ we abbreviate $cl(\{y\})$ as $cl(y)$. A subset of X is called an **F_σ-set** if it is the countable union of closed subsets and a **G_δ-set** if it is the countable intersection of open sets. A subset $N \subseteq X$ is a **neighborhood** (nbd) of $A \subseteq X$ if $A \subseteq int(N)$. We define the set of open neighborhoods of A as $\mathcal{N}(A) := \{G \in \mathcal{G} \mid A \subseteq G\}$ and use $\mathcal{N}(x)$ to abbreviate $\mathcal{N}(\{x\})$. A paving $\mathcal{U}$ is called a **fundamental system of neighborhoods** (fsn) for $A \subseteq X$, if for all $x \in A$ and $G \in \mathcal{N}(x)$, there exists $N \in \mathcal{U}$ with $x \in int(N)$ and $N \subseteq G$. A space X is **first countable** if $\{x\}$ has a countable fsn for every $x \in X$ and **second countable** if X has a countable fsn. A set B is a **base**[1] for a topological space X if every set in $\mathcal{G}$ is a union of sets in B. A set S is

[1] We reserve the term *basis* for basis in dcpos.

a **subbase** for X if the paving of finite intersections of sets in S is a base for X. A space is second countable if and only if it has a countable base. For any set X and a paving $\mathcal{P}$ on X there exists a unique minimum topology $\mathcal{T}(\mathcal{P})$ on X which has $\mathcal{P}$ as a subbase; we call it the **topology generated** by $\mathcal{P}$. The paving $\mathcal{T}(\mathcal{P})$ consists of $\emptyset$, X, and arbitrary unions of finite intersections of sets in $\mathcal{P}$. If $\mathcal{G}$ and $\mathcal{H}$ are topologies on X, then their **join** $\mathcal{G} \vee \mathcal{H}$ is defined as $\mathcal{T}(\mathcal{G} \cup \mathcal{H})$, that is, the coarsest topology that contains $\mathcal{G}$ and $\mathcal{H}$. If $Y \subseteq X$ then the trace $tr(\mathcal{G} \mid Y)$ (also denoted as $\mathcal{G} \cap Y$ or $\mathcal{G} \restriction_Y$) is a topology on Y which we call the **induced** topology on Y. We say that $(Y, \mathcal{G} \cap Y)$ is a **subspace** of $(X, \mathcal{G})$. Suppose that $(Y, \mathcal{H})$ is a topological space. If there exists a continuous injective function $e : Y \to X$ such that $(e[Y], \mathcal{G} \cap e[Y])$ is homeomorphic to $(Y, \mathcal{H})$, then e is called a **subspace embedding** from Y into X. If such an embedding exists, then as far as topological properties are concerned, it is as if $Y \subseteq X$. The following remark has a trivial proof but will be very useful.

Remark 1.4 *Suppose that X is any set, $Y \subseteq X$ and $\mathcal{P}$ is a paving on X. Then*

1. *The topology induced by $\mathcal{T}(\mathcal{P})$ on Y is the same as the topology on Y generated by the paving $\mathcal{P} \cap Y$. That is, $\mathcal{T}(\mathcal{P}) \restriction_Y = \mathcal{T}(\mathcal{P} \cap Y)$.*
2. *If $Z \subseteq Y$, then $(\mathcal{T}(\mathcal{P}) \restriction_Y) \restriction_Z = \mathcal{T}(\mathcal{P} \cap Z)$.*

Proof

1. Clearly $\mathcal{T}(\mathcal{P} \cap Y) \subseteq \mathcal{T}(\mathcal{P}) \restriction_Y$ since $\mathcal{P} \cap Y \subseteq \mathcal{T}(\mathcal{P}) \restriction_Y$. Conversely suppose that $G \in \mathcal{T}(\mathcal{P}) \restriction_Y$. Then

$$\begin{aligned} G &= Y \cap G_2 && \text{for some } G_2 \in \mathcal{T}(\mathcal{P}) \\ &= Y \cap \bigcup_{j \in J} \bigcap_{k \in N_j} G_{(j,k)} && \text{with } G_{(j,k)} \in \mathcal{P},\ N_j \text{ finite} \\ &= \bigcup_{j \in J} \bigcap_{k \in N_j} (Y \cap G_{(j,k)}) \in \mathcal{T}(\mathcal{P} \cap Y). \end{aligned}$$

2. By the first part $(\mathcal{T}(\mathcal{P}) \restriction_Y) \restriction_Z = \mathcal{T}(\mathcal{P} \cap Y) \restriction_Z = \mathcal{T}((\mathcal{P} \cap Y) \cap Z)$ and $(\mathcal{P} \cap Y) \cap Z = \mathcal{P} \cap Z$ since $Z \subseteq Y$. ■

Suppose that $(X, \mathcal{G})$ is a topological space and $x, y \in X$ with $x \neq y$. We have the following separation axioms:

- X is T_0 if there exists a nbd of x that does not contain y or vice versa.
- X is T_1 if there exists $G_x \in \mathcal{N}(x)$ and $G_y \in \mathcal{N}(y)$ with $x \notin G_y$ and $y \notin G_x$.

- X is T_2 if x and y have disjoint nbds, i.e. X is T_1 and G_x and G_y can be chosen such that $G_x \cap G_y = \emptyset$.
- X is **regular** if for all $x \in X$ and closed sets $F \subseteq X$, $x \notin F$ implies that there exist open sets G_x and G_F such that $x \in G_x$, $F \subseteq G_F$ and $G_x \cap G_F = \emptyset$. This is equivalent to saying that $\mathcal{F}$ is a fundamental system of neighborhoods for X.
- X is **completely regular** if for all $x \in X$ and closed sets $F \subseteq X$, $x \notin F$ implies that there exists a continuous map (Urysohn function) $f : X \to [0,1]$ such that $f(x) = 0$ and $f(y) = 1$ for all $y \in F$.
- X is **normal** if any two disjoint closed subsets have disjoint nbds.

T_2 spaces are called **Hausdorff** spaces. We have that $T_2 \Rightarrow T_1 \Rightarrow T_0$ and that complete regularity $\Rightarrow$ regularity. Nevertheless regularity $\not\Rightarrow T_2$, in fact, regularity does not even imply T_0. This gives rise to the following definitions: a space that is regular and T_0 is called T_3; a space that is completely regular and T_0 is $T_{3\frac{1}{2}}$ or **Tychonoff**; a space that is normal and T_1 is T_4. It can be shown that $T_4 \Rightarrow T_{3\frac{1}{2}} \Rightarrow T_3 \Rightarrow T_2 \Rightarrow T_1 \Rightarrow T_0$.

Remark 1.5 *Unfortunately there exists an annoying lack of consensus among topologists regarding the notation for the separating axioms. Some authors (e.g. Bourbaki [Bou66], Steen and Seebach [SS78]) use our definition of regularity as the definition of T_3. Under this convention $T_3 \not\Rightarrow T_2$ and they define regularity as T_3 plus T_0. Other authors (e.g. Dugundji [Dug66]) require T_2 as part of their definition of regularity, hence T_3 and regularity become synonyms. Our definition follows the classic textbook by Kelley [Kel55]. Similar remarks apply for complete regularity and normality.*

A detailed discussion of these and other axioms can be found in [SS78, Section 2]. A Hausdorff space X is called **separable** if it contains a countable dense set. That is, there exists $D \subseteq X$ such that D is countable and $cl(D) = X$. A space X is **Lindelöf** if every open covering for X (that is, a $\mathcal{G}$-covering) admits a countable subcovering. X is **fully Lindelöf** if every family $\mathcal{G}^* \subseteq \mathcal{G}$ contains a countable subfamily with the same union; this is equivalent to saying that every subspace of X is Lindelöf, hence these spaces are also called **hereditarily Lindelöf**. It is easy to show that second countable spaces are fully Lindelöf. X is **compact** if it satisfies the Heine-Borel property, i.e. every open covering for X admits a finite subcovering. A subset K of X is said to be **compact** if the subspace $(K, tr(\mathcal{G} \mid K))$ is compact. Bourbaki [Bou66, I, 9, 1] prefers to call these sets **quasicompact** and reserve the term compact for Hausdorff spaces. Dugundji defines compactness for Hausdorff spaces only. We do not make such distinctions. We denote by $\mathcal{K}$ the paving of all compact subsets of X. X is **locally compact** if $\mathcal{K}$ is a fundamental system of neighborhoods for X. It should be noted that without the assumption of the space being Hausdorff, X is compact does not imply X is locally compact; $\mathcal{K}$ is not necessarily $(\cap f)$-stable nor a subset of $\mathcal{F}$. Let B be a subset of X, we say that B is **relatively compact** if $cl(B)$ is compact and that B is **σ-compact** if B is the countable union of compact sets.

Lemma 1.6 *(Alexander) Let $(X, \mathcal{G})$ be a topological space and $\mathcal{S}$ be a subbase for $\mathcal{G}$. Then $K \subseteq X$ is compact if and only if every $\mathcal{S}$-covering for K has a finite subcovering.*

Proof Alexander's Subbase Lemma (as in [Kel55, Theorem 5.6] for example) states that X is compact if and only if every $\mathcal{S}$-covering for X has a finite subcovering. The conclusion follows from part (1) of Remark 1.4 by considering K with the subspace topology $\mathcal{G} \restriction_K$. ■

Let $(X_j, \mathcal{G}_j)_{j \in I}$ be a family of topological spaces. We define the **Cartesian product** topology $\otimes_j \mathcal{G}_j$ on the set $\Pi_j X_j$ as the minimum topology such that for each $j \in I$ the projection π_j is continuous. Some textbooks denote the product topology as $\Pi_j \mathcal{G}_j$; so it becomes unclear whether one is referring to the product topology or to the Cartesian product of the topologies. We, on the other hand, reserve the symbol Π for the Cartesian product of sets; nevertheless the symbol $\otimes$ is also used to denote the product σ-algebra (see Section 1.8 below). Suppose that $\{j_1, j_2, ..., j_n\} \subseteq_f I$ and $G_k \in \mathcal{G}_{j_k}$ for $k = 1, ..., n$, let

$$\langle G_1, G_2, \ldots, G_n \rangle := \pi_{j_1}^{-1} G_1 \cap \cdots \cap \pi_{j_n}^{-1} G_n$$

That is, $\langle G_1, G_2, ..., G_n \rangle = \Pi_j Y_j$ where

$$Y_j := \begin{cases} G_k & \text{if } j = j_k \text{ for } k = 1, \ldots, n \\ X_j & \text{otherwise.} \end{cases}$$

Hence the set $\{\langle G_1, G_2, ..., G_n \rangle \mid \{j_1, j_2, ..., j_n\} \subseteq_f I, G_k \in \mathcal{G}_{j_k} \text{ for } k = 1, ..., n\}$ is a base for $\otimes_j \mathcal{G}_j$. It follows that the countable product of second countable spaces is second countable. The following result is important.

Proposition 1.7 *(Tychonoff) Let $(X_j, \mathcal{G}_j)_{j \in I}$ be a family of topological spaces, then $(\Pi_j X_j, \otimes_j \mathcal{G}_j)$ is compact if and only if $(X_j, \mathcal{G}_j)$ is compact for all $j \in I$.*

Proof See Theorem I-(9.5)-3 in [Bou66]. ■

A closed nonempty subset F of X is called **irreducible**, if for any closed subsets F_1 and F_2 with $F \subseteq F_1 \cup F_2$, we have $F \subseteq F_1$ or $F \subseteq F_2$. We say X is **sober** if all closed irreducible subsets of X are the closure of a unique point. Hausdorff spaces are sober but there exist examples of T_1 spaces which are not sober [GHK$^+$80, Ex. I-3.36]. For any set $A \subseteq X$ we define the **saturation** of A as $sat(A) := \bigcap\{G \mid G \in \mathcal{G}, A \subseteq G\}$. A set A is said to be **saturated** if $A = sat(A)$. The paving of compact saturated sets will be denoted by $\mathcal{Q}$. Obviously $\mathcal{Q} \subseteq \mathcal{K}$ and for T_1 spaces they coincide. We also have that $Q \in \mathcal{K}$ implies that $sat(Q) \in \mathcal{Q}$. The **cocompact** topology $\mathcal{G}_{cc}$ on X (the de Groot dual of $\mathcal{G}$ [dG67][dGHSW69]) is defined as the topology that has as a base the complements of sets in $\mathcal{Q}$. The **patch** topology $\mathcal{G}_p$ on X is defined as $\mathcal{G} \vee \mathcal{G}_{cc}$. Since $\mathcal{G}$ is $(\cap f)$-stable and $\mathcal{Q}$ is $(\cup f)$-stable, then the set $B_p := \{G \cap Q^{\mathsf{c}} \mid G \in \mathcal{G}, Q \in \mathcal{Q}\}$ is a base for the patch topology. $(X, \mathcal{G}_p)$ is called the **patch space** of $(X, \mathcal{G})$. For $\bullet = cc, p$ we set $X^\bullet := (X, \mathcal{G}_\bullet)$.

1.5 Nets

Let $(J, \sqsubseteq)$ be a directed set and Ω be any set. A **net** $(x_j)_{j\in J}$ in Ω is a function $j \mapsto x_j : J \to \Omega$. If $B \subseteq \Omega$ we say that $(x_j)_{j\in J}$ is **eventually** in B if there exists $k \in J$ such that $x_j \in B$ whenever $k \sqsubseteq j$. If P is a property of the elements of Ω we say that $P(x_j)$ holds eventually if $(x_j)_{j\in J}$ is eventually in $\{x \in \Omega \mid P(x)\}$. If $(\Omega, \leq)$ is a poset and $j \sqsubseteq k$ implies that $x_j \leq x_k$ we say that the net is **monotone** [GHK$^+$80, Definition 0-1.2]. Monotone nets are denoted as $\langle x_j \rangle_{j\in J}$. Notice that the image of a monotone net is a directed subset of Ω. Conversely, for any directed subset $B \subseteq \Omega$ the inclusion map is a monotone net. Hence monotone nets and directed subsets will be used interchangeably. When the domain of a monotone net is not specified it is understood to be a directed set of indexes with a partial order; alternatively we write $\langle x_j \rangle_{j\in(J,\sqsubseteq)}$ to stress that J is ordered by $\sqsubseteq$. Sequences and monotone sequences in $\mathbb{R}$ are particular examples of nets and monotone nets respectively. A net $(x_j)_{j\in J}$ in a topological space $(X, \mathcal{G})$ **converges** to $x \in X$ if it is eventually in each neighborhood of x; this is denoted $(x_j) \to x$. A net $(x_{j_k})_{k\in(I,\leq)}$ is a **subnet** of $(x_j)_{j\in J}$ if $\{j_k \mid k \in I\} \subseteq J$ and for any $j \in J$ the net $(j_k)_{k\in I}$ is eventually in $\uparrow j$. A set $C \subseteq J$ is called **cofinal** in J if for all $j \in J$ there exists $c \in C$ with $j \sqsubseteq c$.

Remark 1.8 ***An important difference between nets and sequences.*** *If $C \subseteq J$ is cofinal, then $(C, \sqsubseteq)$ is directed and $(x_j)_{j\in C}$ is a subnet of $(x_j)_{j\in J}$.*

Conversely, if $(x_{j_k})_{k\in I}$ is a subnet of $(x_j)_{j\in J}$, then the set $D := \{j_k \mid k \in I\}$ is cofinal in J and obviously $\{x_{j_k} \mid k \in I\} = \{x_j \mid j \in D\}$. Nevertheless $(x_{j_k})_{k\in(I,\leq)}$ and $(x_j)_{j\in(D,\sqsubseteq)}$ are not in general the same subnet. The difference lies in the order: for example, $(J,\sqsubseteq)$ (and hence $(D,\sqsubseteq)$) could be a totally ordered set but this need not be the case for $(I,\leq)$. In fact there exist examples where $(x_{j_k})_{k\in(I,\leq)}$ converges to a point but $(x_j)_{j\in(D,\sqsubseteq)}$ does not (see Exercise 2.E in [Kel55]).

Therefore the subnets arising from cofinal sets with the inherited order are not enough to ensure the convergence property for compact sets in the next proposition.

Proposition 1.9 *Let $(X,\mathcal{G})$ and $(Y,\mathcal{H})$ be topological spaces. Then*

1. *x is an accumulation point of a subset $B \subseteq X$ if and only if there exists a net $(x_j)_{j\in J}$ in B such that $(x_j) \to x$.*
2. *$F \in \mathcal{F}X$ if and only if for every net $(x_j)_{j\in J}$ in F such that $(x_j) \to x$, then $x \in F$.*
3. *$K \in \mathcal{K}X$ if and only if every net in K has a subnet that converges to some point in K.*
4. *A function $f : X \to Y$ is continuous if and only if for each net $(x_j)_{j\in J}$ in X such that $(x_j) \to x$, we have that $(f(x_j)) \to f(x)$.*

Proof See Theorems 2-2, 3-1 and 5-2 in [Kel55]. ■

Suppose that $(L,\leq)$ is a complete lattice and $(x_j)_{j\in J}$ is a net in L. We define [GHK+80, Definition II-1.1]

$$\begin{aligned} \liminf_j x_j &= \textstyle\bigvee^{\uparrow}_{j\in J} \bigwedge\{x_i \mid j \sqsubseteq i\} \\ \limsup_j x_j &= \textstyle\bigwedge_{j\in J} \bigvee\{x_i \mid j \sqsubseteq i\}. \end{aligned}$$

If $\liminf_j x_j = \limsup_j x_j = x$, then we write $x = \lim_j x_j$. Is not difficult to show that for every net $(x_j)_{j\in J}$ in $\overline{\mathbb{R}}$, we have that $(x_j) \to x$ if and only if $x = \lim_j x_j$. If $(x_j)_{j\in J}$ is a net in a metric space (X,d) (see Section 1.12 below), then $(x_j) \to x$ if for all $\varepsilon > 0$ there exists $k \in J$ such that $d(x_j, x) < \varepsilon$ when $k \sqsubseteq j$. The net $(x_j)_{j\in J}$ is a **Cauchy net** if for all $\varepsilon > 0$ there exists $k \in J$ such that $d(x_n, x_m) < \varepsilon$ when $k \sqsubseteq n, m$.

1.6 Order related topologies

Let $(P,\sqsubseteq)$ be a poset. The **Alexandroff** topology [Ale37] $\mathcal{G}_A$ on P is the paving of all upper sets. That is, the topology with $\{\uparrow x \mid x \in P\}$ as a basis.

The **Scott topology** on P is defined as follows: $G \subseteq P$ is open if G is upper and for all directed subsets $D \subseteq P$ such that $\bigsqcup^{\uparrow} D$ exists, $\bigsqcup^{\uparrow} D \in G$ implies that $G \cap D \neq \emptyset$. The Scott topology is denoted by $\mathcal{G}_\sigma$. The **upper** topology $\mathcal{G}_u$ on P is defined as the topology that has as a subbasis $\{(\downarrow x)^c \mid x \in P\}$. Since $F \subseteq_f P$ implies that $\downarrow F$ is Scott closed, it follows that $\mathcal{G}_u \subseteq \mathcal{G}_\sigma$. A topology $\mathcal{G}$ on a dcpo $(D, \sqsubseteq)$ is said to be **order consistent** [GHK+80, Exercise II-1.16] if $\mathcal{G}_u \subseteq \mathcal{G} \subseteq \mathcal{G}_\sigma$. The **lower** topology $\mathcal{G}_l$ on P is defined as the topology that has as a subbasis $\{(\uparrow x)^c \mid x \in P\}$. The **Lawson** topology $\mathcal{G}_\lambda$ on P is defined as $\mathcal{G}_\sigma \vee \mathcal{G}_l$. For $\bullet = \sigma, l, \lambda, A$ we set $P^\bullet := (P, \mathcal{G}_\bullet P)$.

Proposition 1.10 *Let $(P, \sqsubseteq)$ be a continuous poset. Then the set $\twoheaduparrow C$ is Scott open for all $C \subseteq P$. The space P^σ is sober if and only if P is a dcpo. If P is a continuous dcpo, then*

1. *P^σ is a locally compact sober space.*
2. *$G = \bigcup_{x \in G} \twoheaduparrow x = \bigcup_{x \in G} \uparrow x$, for all $G \in \mathcal{G}_\sigma P$. That is, $\{\twoheaduparrow x \mid x \in P\}$ is a base for the Scott topology.*
3. *The cocompact and patch topologies on P^σ agree with the lower $\mathcal{G}_l P$ and Lawson $\mathcal{G}_\lambda P$ topologies respectively.*

Proof See Corollaries 3.12, 4.19 and Lemma 4.16 in [Mis98]; Section V in [Law88]. ■

Suppose that $C \subseteq \overline{\mathbb{R}}$ is a closed interval (of possibly infinite length). Then $(C, \leq)$ is a continuous dcpo where $\mathcal{G}_\sigma C = \{C \cap (a, \infty] \mid a \in \overline{\mathbb{R}}\}$, $\mathcal{G}_l C = \{C \cap [-\infty, b) \mid b \in \overline{\mathbb{R}}\}$ and $\mathcal{G}_\lambda C$ is the metric topology on C. We define $C^{\leq} := (C, \mathcal{G}_\sigma C)$ and $C^{\geq} := (C, \mathcal{G}_l C)$. The reason for introducing these definitions is that $(C, \leq^{op})$ is also a poset and the Scott topology on $(C, \leq^{op})$ is the lower topology on $(C, \leq)$ and vice versa. Hence, referring to C^σ could generate ambiguity; on the other hand C^λ is always C endowed with the metric topology. As a consequence of these definitions for a topological space $(Y, \mathcal{H})$ we have that $[Y \to \mathbb{R}^{\leq}]$, $[Y \to \mathbb{R}^{\geq}]$, $[Y \to \mathbb{R}^\lambda]$ are the sets of **lower semi-continuous** (LSC), **upper semi-continuous** (USC) and continuous functions from Y to $\mathbb{R}$ respectively.

Every T_0 topological space $(X, \mathcal{G})$ carries an inherent partial order called the **specialisation order** given by: $x \sqsubseteq_s y$ if and only if $x \in cl(y)$. If X is not a T_0 space, then the specialisation relation is only a preorder. A set is saturated if and only if it is an upper set with respect to the specialization preorder (see part (i) of Lemma 3.41). Notice that on a dcpo $(D, \sqsubseteq)$ with the Scott topology we have that $cl(y) = \downarrow y$ for all $y \in D$, hence the specialisation order agrees with $\sqsubseteq$. Let $(X, \mathcal{G})$ be T_0 space, we say that X is a **monotone convergence space** (mc-space) [GHK+80, Definition II-3.9] or ***d*-space**[2] if $(X, \sqsubseteq_s)$ is a dcpo and $\mathcal{G} \subseteq \mathcal{G}_\sigma$, i.e. $\mathcal{G}$ is coarser than the Scott topology. In a monotone convergence space every directed set D converges to its least upper bound. That is, for all $G \in \mathcal{G}$ such that $\bigsqcup^\uparrow D \in G$ there exists $x \in D$ such that for all $y \in X$ if $x \sqsubseteq_s y$, then $y \in D$. Notice also that if X is an mc-space, then for all $x \in X$ we have that $cl(x) = \downarrow x$; therefore $\mathcal{G}_u \subseteq \mathcal{G}$. Hence $(X, \mathcal{G})$ is a mc-space if and only if $(X, \sqsubseteq_s)$ is a dcpo and the topology $\mathcal{G}$ is order consistent with it. Dcpo's (with the Scott topology) are in particular monotone convergence spaces since (as we observed) their order agrees with the specialisation order. Sober spaces are also monotone convergence spaces (Proposition 3.9). If $(D_j, \sqsubseteq_j)_{j \in I}$ is a family of dcpo's and $\mathcal{G}^j_\sigma$ is the Scott topology in D_j, then the Cartesian product $(\Pi_j D_j, \otimes_j \mathcal{G}^j_\sigma)$ is a monotone convergence space; therefore the product topology $\otimes_j \mathcal{G}^j_\sigma$ is a subset of the Scott topology of $(\Pi_j D_j, \sqsubseteq)$. If the D_j are continuous and I is finite, then the Scott topology and the product topology agree; if I is infinite, the result still holds if the D_j are continuous and all except a finite number of them have a minimum element.

Suppose that $(X, \mathcal{G})$ is a topological space equipped with a partial order $\leq$. We say that X is a **pospace**[3] [Nac65] if $R_\leq$ is a closed subset of $X \times X$ with respect to the product topology $\mathcal{G} \otimes \mathcal{G}$. Pospaces are always Hausdorff. If X is compact, then $(X, \mathcal{G}, \leq)$ is called a **compact pospace**. The following definition is important since there is no standard agreement in the area for some of these terms. Our terminology agrees with that of Smyth [Smy92a] and Erker *et al.* [EEK98]

Definition 1.11 *Let $(X, \mathcal{G})$ be a topological space.*

- *X is **coherent** if it is sober and $\mathcal{Q}$ is $(\cap f)$-stable (we do not assume that X is compact). If X is also a continuous dcpo, then it is called a **coherent domain**.*
- *X is **supersober** or strongly sober if the set of limit points of every ultrafilter is either empty or the closure of a unique point.*
- *X is **stably locally compact** if it is sober, locally compact and for all $G_1, G_2, H \in \mathcal{G}$ if $H \ll_{\mathcal{G}} G_1$ and $H \ll_{\mathcal{G}} G_2$, then $H \ll_{\mathcal{G}} G_1 \cap G_2$, i.e.*

[2] See [Ers99] for an in-depth account.

[3] Originally, the term pospace abbreviates "partially ordered space", nevertheless we reserve the latter name for any topological space (not necessarily Hausdorff) where a partial order is defined.

the way below relation on $\mathcal{G}$ is multiplicative.

- *X is* ***stably compact*** *if it is compact and stably locally compact. If X is also a continuous dcpo, then it is called a* ***stably compact domain****.*

Supersober spaces are sober [GHK+80, Proposition VII-1.11]. Recently A. Jung has proposed the name **Nachbin** spaces [Nac65] for stably compact spaces.

Proposition 1.12 *Let $(X, \mathcal{G})$ be a locally compact topological space. Then the following conditions are equivalent:*

1. *X is supersober.*
2. *X is stably locally compact.*
3. *X is coherent.*

Proof See pages 302-303 in [HL78], Corollary V-5.13 and Exercise VII-1.22 in [GHK+80] and Theorem 4.8 in [HM81]. See also Theorem 4.11 in [Kop95] for another characterisation on bitopological spaces[4]. ■

If $(X, \mathcal{H}, \leq)$ is a pospace, we define

$$\mathcal{U}_{\leq} := \{G \in \mathcal{H} \mid G = \uparrow G\}$$
$$\mathcal{L}_{\leq} := \{G \in \mathcal{H} \mid G = \downarrow G\}.$$

That is, $\mathcal{U}_{\leq}$ is the set of $\leq$-upper open sets and $\mathcal{L}_{\leq}$ is the set of $\leq$-lower open sets. Recall the following equivalence between stably compact spaces and compact pospaces.

Proposition 1.13

1. *Let $(X, \mathcal{G})$ be a stably compact space and $\sqsubseteq_s$ its specialisation order. Then $(X, \mathcal{G}_p, \sqsubseteq_s)$ is a compact pospace where the open upper sets are the sets in $\mathcal{G}$ and the open lower sets are the sets in $\mathcal{G}_{cc}$.*
2. *Let $(X, \mathcal{H}, \leq)$ be a compact pospace. Then $(X, \mathcal{U}_{\leq})$ is a stably compact space where the specialisation order coincides with $\leq$, the patch topology $(\mathcal{U}_{\leq})_p = \mathcal{H}$ and the cocompact topology $(\mathcal{U}_{\leq})_{cc} = \mathcal{L}_{\leq}$.*

Proof See Chapter VII, exercises 1.18, 1.19, 1.22 in [GHK+80]. See also Corollary 20 and Theorem 25 in [Law91] and Theorem 20 in [Smy91] for a further equivalence with complete totally bounded quasi-uniform spaces. See

[4] Kopperman and Lawson's definition of supersober is stronger than ours since they do not allow the set of limit points to be empty.

Theorem 5.3 in [Kop95] for another equivalence on bitopological spaces. ■

1.7 Measure theory

Our main references in probability and measure theory are [Top70b][Top78]-[HJ94]; the reader is also referred to [Bil86][Hal50][Var65]. Let Ω be any set. A $(\cap f)$-paving $\mathcal{P}$ on Ω is called a π-**system**. Let $\mathcal{S}$ be a $(\emptyset, \Omega, \cap f)$-paving on Ω. $\mathcal{S}$ is a **Boolean semialgebra** of subsets of Ω if for all $s \in \mathcal{S}$ there exists a finite subset $\{s_i\}_{i=1}^n$ of $\mathcal{S}$ such that $s^{\mathsf{c}} = \sum_{i=1}^n s_i$. i.e. the complement of a set in $\mathcal{S}$ can be expressed as a finite disjoint union of sets in $\mathcal{S}$. A **Boolean algebra** of subsets of Ω is a $(\emptyset, C, \cap f)$-paving on Ω. A $\boldsymbol{\sigma}$**-algebra** of subsets of Ω is a $(\emptyset, C, \cap c)$-paving on Ω. If $\mathcal{P}$ is a paving on Ω it is possible to show that there exist a minimum algebra $\mathcal{A}(\mathcal{P})$ and a minimum σ-algebra $\sigma(\mathcal{P})$ on Ω that contain $\mathcal{P}$. We call them the algebra and the σ-algebra generated by $\mathcal{P}$ respectively. If $\mathcal{S}$ is a semialgebra, then $\mathcal{A}(\mathcal{S}) := \{\sum_{i=1}^n s_i \mid s_i \in \mathcal{S}$ and $n \in \mathbb{N}\}$. The **Borel** σ**-algebra** $\mathcal{B}X$ of a topological space $(X, \mathcal{G})$ is defined as $\sigma(\mathcal{G})$. Let $\mathcal{P}$ be a $(\emptyset)$-paving, $\mathcal{A}$ be an algebra and Γ be a σ-algebra on Ω, then

- A function $\mu : \mathcal{P} \to \overline{\mathbb{R}}_+$ is called a **set function** if it satisfies $\mu(\emptyset) = 0$ (strictness).
- A set function $\mu : \mathcal{A} \to \overline{\mathbb{R}}_+$ is called a **content**[5] if $A_1 \cap A_2 = \emptyset$ implies that $\mu(A_1 \cup A_2) = \mu(A_1) + \mu(A_2)$ (finite additivity).
- A content $\mu : \Gamma \to \overline{\mathbb{R}}_+$ is called a **measure** if $\mu(\bigcup_{n\in\mathbb{N}} E_n) = \sum_{n=0}^\infty \mu(E_n)$ provided that $E_k \cap E_j = \emptyset$ for all $k \neq j$ (countable additivity).
- A set function $\mu : \mathcal{P} \to \overline{\mathbb{R}}_+$ is σ**-subadditive** if $\mu(\bigcup_{n\in\mathbb{N}} E_n) \leq \sum_{n=0}^\infty \mu(E_n)$ for all sequences (E_n) in $\mathcal{P}$ such that $\bigcup_{n\in\mathbb{N}} E_n \in \mathcal{P}$. Measures are always σ-subadditive.

We also have that for all $\Omega_0 \subseteq \Omega$ the traces $tr(\mathcal{A} \mid \Omega_0)$ and $tr(\Gamma \mid \Omega_0)$ are an algebra and a σ-algebra on Ω_0 respectively [HJ94, Exercise 1.28]. A **measurable space** consists of a pair (Ω, Γ) where Ω is any set and Γ is a σ-algebra of subsets of Ω. A **measure space** is a triple (Ω, Γ, μ) such that (Ω, Γ) is a measurable space and μ is a measure on Γ. A **probability** space is a measure space $(\Omega, \Gamma, \mathbb{P})$ where $\mathbb{P}(\Omega) = 1$. The following result gives criteria for extending a content to a measure.

Proposition 1.14 *Suppose that $\mathcal{A}$ is an algebra on Ω.*

[5] This is normally found in the literature as finitely additive measure, charge, or measure on an algebra.

1. *Let $\mu : \mathcal{A} \to \mathbb{R}_+$ be a content. Then μ is countably additive*[6] *if and only if for every decreasing sequence $(A_n)_{n\in\mathbb{N}}$ in $\mathcal{A}$ such that $(A_n)\downarrow\emptyset$, we have that $\lim_n \mu(A_n) = 0$.*

2. *If $\mu : \mathcal{A} \to \overline{\mathbb{R}}_+$ is a content which is countably additive on $\mathcal{A}$, then it can be extended to a measure on $\sigma(\mathcal{A})$.*

Proof See Theorems 3.1.1 and 3.1.4 in [Dud89]. ∎

For any measure space (Ω, Γ, μ) we introduce the following definitions:

- μ is **finite** if $\mu(\Omega) < \infty$.
- μ is **σ-finite** if there exists a sequence (E_n) in Γ such that $\mu(E_n) < \infty$ for all $n \in \mathbb{N}$ and $\Omega = \bigcup_{n\in\mathbb{N}} E_n$.
- Γ is a **countably generated σ-algebra** if there exists a countable paving $\mathcal{C} \subseteq \Omega$ such that $\Gamma = \sigma(\mathcal{C})$. In general, Γ is **$\aleph_\alpha$-generated** if the paving $\mathcal{C}$ has cardinality $\aleph_\alpha$ for some ordinal α.

If $(X, \mathcal{G})$ is a topological space, then a measure μ on $(X, \mathcal{B}X)$ is called a **Borel measure**; $\mathfrak{M}(X)$ denotes the set of Borel measures on $(X, \mathcal{B}X)$. The following definitions are taken from the Appendix in [THJ80].

- μ is **locally finite** if for all $x \in X$ there exists $N \in \mathcal{N}(x)$ such that $\mu(N) < \infty$.
- μ is **τ-smooth** if for every directed family $\mathcal{G}^* \subseteq \mathcal{G}$ we have that

  $\mu(\bigcup^{\uparrow} \mathcal{G}^*) = \sup_{G\in\mathcal{G}^*} \mu(G)$. We set $\mathfrak{M}(X;\tau) := \{\mu \in \mathfrak{M}(X) \mid \mu$ is τ-smooth$\}$.
- μ is **$\mathcal{F}$-regular** if $\mu(E) = \sup\{\mu(F) \mid F \subseteq E,\ F \in \mathcal{F}\}$ for all $E \in \mathcal{B}X$. We set $\mathfrak{M}(X;F) := \{\mu \in \mathfrak{M}(X) \mid \mu$ is $\mathcal{F}$-regular$\}$.
- μ is **tight** or **inner regular** if $\mu(E) = \sup\{\mu(C) \mid C \subseteq E,\ C \in \mathcal{K}\}$ for all $E \in \mathcal{B}X$. We set $\mathfrak{M}(X;R) := \{\mu \in \mathfrak{M}(X) \mid \mu$ is inner regular$\}$.
- μ is **outer regular** if $\mu(E) = \inf\{\mu(G) \mid E \subseteq G,\ G \in \mathcal{G}\}$ for all $E \in \mathcal{B}X$.
- μ is **regular** if μ is inner regular and outer regular.
- μ is a **Radon** measure if μ is inner regular and locally finite.

We also define $\mathfrak{M}_+(X) = \{\mu \in \mathfrak{M}(X) \mid \mu(X) < \infty\}$, $\mathfrak{M}^r(X) = \{\mu \in \mathfrak{M}(X) \mid \mu(X) = r\}$ for $r \in (0, \infty)$, $\mathfrak{M}^{\leq 1}(X) = \{\mu \in \mathfrak{M}(X) \mid \mu(X) \leq 1\}$.With

[6] With respect to disjoint sequences (E_n) of sets in $\mathcal{A}$ such that $\bigcup_n E_n \in \mathcal{A}$.

the obvious meaning for the possible combinations $\mathfrak{M}^1(X;\tau) = \mathfrak{M}(X;\tau) \cap \mathfrak{M}^1(X)$, etc.

Remark 1.15 *In [Top70b] and [Top78] τ-smoothness is defined in a weaker way which for finite measures is equivalent to our definition; $\mathcal{F}$-regularity is only called regularity which generates confusion with respect to the classic definition presented here.*

It is easy to show that

$$\left.\begin{array}{l}\text{Radon} \Rightarrow \\ \text{regular} \Rightarrow\end{array}\right\} \text{inner regular} \Rightarrow \tau\text{-smooth}$$

and that in fully Lindelöf spaces (e.g. second countable spaces) every measure is τ-smooth. If μ is finite, then $\mathcal{F}$-regularity and outer regularity are equivalent. Propositions 1.28 and 1.29 below give more information for Hausdorff spaces. Let (Ω_1, Γ_1) and (Ω_2, Γ_2) be measurable spaces. A function $f : \Omega_1 \to \Omega_2$ is **measurable** if $f^{-1}E \in \Gamma_1$ for all $E \in \Gamma_2$. The set $\sigma(f) := \{f^{-1}E \mid E \in \Gamma_2\}$ is a σ-algebra on Ω_1 which we call the σ-algebra **induced** by f on Ω_1. Hence f is measurable if $\sigma(f) \subseteq \Gamma_1$. In the same way if $\{f_i\}_{i\in I}$ is a family of functions $f_i : \Omega \to \Omega_i$ where (Ω_i, Γ_i) are measurable spaces for all $i \in I$, then $\sigma(f_i : i \in I)$ is the minimum σ-algebra on Ω which makes all the functions f_i measurable. Let $U \subseteq \Omega$, we define the **indicator function** $\mathbb{I}_U : \Omega \to [0,1]$ as:

$$\mathbb{I}_U(x) := \begin{cases} 1 & \text{if } x \in U \\ 0 & \text{otherwise} \end{cases}$$

If (Ω, Γ) is a measurable space then $\mathbb{I}_U$ is measurable if and only if $U \in \Gamma$. If $(X, \mathcal{G})$ is a topological space and $[0,1]$ is endowed with its usual topology or with the Scott topology, then $\mathbb{I}_U$ is continuous if and only if $U \in \mathcal{G}$. A **simple function** is a finite linear combination $\sum_{i=1}^n r_i \mathbb{I}_{Ui}$ of indicator functions with $r_i \in (0, \infty)$ for $i = 1,..., n$.

Suppose that $\eta : 2^\Omega \to \overline{\mathbb{R}}_+$ is a set function. A subset E of Ω is called η-**measurable** if:

$$\eta(A) = \eta(A \cap E) + \eta(A \backslash E) \text{ for all } A \subseteq \Omega$$

The paving of all η-measurable subsets is denoted by $\mathcal{M}(\eta)$ and we have the following result:

Proposition 1.16 *(Carathéodory) Let $\eta : 2^\Omega \to \overline{\mathbb{R}}_+$ be a set function. Then*

$\mathcal{M}(\eta)$ is an algebra and η is a content on $\mathcal{M}(\eta)$. If η is σ-subadditive then $\mathcal{M}(\eta)$ is a σ-algebra and η is a measure on $\mathcal{M}(\eta)$.

Proof See [HJ94, Section 1.23]. ■

Let $\mathcal{P}$ be a $(\emptyset)$-paving on Ω and $\mu : \mathcal{P} \to \overline{\mathbb{R}}_+$ be a set function.

- The **inner μ-measure** [HJ94, Sec. 1.22] $\mu_* : 2^\Omega \to \overline{\mathbb{R}}_+$ is defined as

$$\mu_*(F) := \sup\{\textstyle\sum_{j=1}^n \mu(E_j) \mid E_1, \dots, E_n \text{ disjoint} \in \mathcal{P} \text{ and } \sum_{j=1}^n E_j \subseteq F\}.$$

- The **outer μ-measure** $\mu^* : 2^\Omega \to \overline{\mathbb{R}}_+$ is defined as:

$$\mu^*(A) := \inf\{\textstyle\sum_{n\in\mathbb{N}} \mu(E_n) \mid A \subseteq \bigcup_{n\in\mathbb{N}} E_n \text{ and } E_n \in \mathcal{P} \text{ for all } n \in \mathbb{N}\}.$$

If μ is a content on an algebra $\mathcal{A}$, then $\mu_* \restriction_{\mathcal{A}} = \mu$ and $\mu_*(B) = \sup\{\mu(E) \mid E \subseteq B, E \in \mathcal{A}\}$ for all $B \subseteq \Omega$. If μ is a measure on a σ-algebra Γ then $\mu^* \restriction_\Gamma = \mu$ and $\mu^*(B) = \inf\{\mu(E) \mid B \subseteq E, E \in \Gamma\}$ (see [HJ94, Exercises 1.31, 1.32]). If (Ω, Γ, μ) is a measure space, then it is possible to show that μ^* is σ-subadditive on 2^Ω and $\Gamma \subseteq \mathcal{M}(\mu^*)$. By Proposition 1.16 we have that $\mathcal{M}(\mu^*)$ is a σ-algebra and μ^* is a measure on $\mathcal{M}(\mu^*)$ that extends μ. We define $\mathcal{M}(\mu) := \mathcal{M}(\mu^*)$ and $\widehat{\mu} := \mu^* \restriction_{\mathcal{M}(\mu^*)}$. Then the measure space $(\Omega, \mathcal{M}(\mu), \widehat{\mu})$ is called the **completion** of μ under Γ. Elements of $\mathcal{M}(\mu)$ are called **μ-measurable** sets. The measure space (Ω, Γ, μ) is said to be **complete** if $\Gamma = \mathcal{M}(\mu)$ and $\mu = \widehat{\mu}$.

Define the paving

$$\begin{aligned} \mathcal{L}(\mu) &:= \{B \subseteq \Omega \mid \exists\, E_0, E_1 \in \Gamma \text{ with } E_0 \subseteq B \subseteq E_1 \text{ and } \mu(E_1 \backslash E_0) = 0\} \\ &= \{E \cup B \mid E \in \Gamma \text{ and exists } N \in \Gamma \text{ with } B \subseteq N \text{ and } \mu(N) = 0\} \end{aligned}$$

Then $\mathcal{L}(\mu)$ is a σ-algebra and its elements are called **Lebesgue μ-measurable** subsets of Ω.

Proposition 1.17 *Let (Ω, Γ, μ) be a measure space. Then*

1. *$\Gamma \subseteq \mathcal{L}(\mu) \subseteq \mathcal{M}(\mu)$.*
2. *If $\mu_*(B) = \mu^*(B) < \infty$ then $B \in \mathcal{L}(\mu)$.*
3. *$\mu_*(B) = \mu^*(B)$ for all $B \in \mathcal{L}(\mu)$.*
4. *If μ is σ-finite then $\mathcal{L}(\mu) = \mathcal{M}(\mu)$. In particular this is true for probability measures.*

Proof See Section 1.28 in [HJ94]. ■

Let (Ω, Γ) be a measurable space. A set $B \subseteq \Omega$ is **universally measurable** if B is measurable with respect to every finite (equiv. σ-finite) measure on (Ω, Γ). The paving of universally measurable subsets of (Ω, Γ) is denoted by $\mathcal{U}(\Gamma)$. Since $\mathcal{U}(\Gamma) = \bigcap\{\mathcal{M}(\mu) \mid \mu \text{ is a measure on } (\Omega, \Gamma)\}$ then $\mathcal{U}(\Gamma)$ is a σ-algebra and $\Gamma \subseteq \mathcal{U}(\Gamma)$.

In many situations it is desirable to restrict or extend a measure in a consistent way to a set which does not belong to the σ-algebra where the measure is defined. This can be done as follows: let (Ω, Γ, μ) be a measure space and $\Omega_0 \subseteq \Omega$ be arbitrary. Let $\Gamma \restriction_{\Omega_0}$ be the trace $tr(\Gamma \mid \Omega_0) = \{E \cap \Omega_0 \mid E \in \Gamma\}$. Thus $\Gamma \restriction_{\Omega_0}$ is a σ-algebra on Ω_0 and the following statements hold:

Proposition 1.18 *Suppose that (Ω, Γ, μ) is a measure space and $\Omega_0 \subseteq \Omega$ is any subset. Then,*

1. *μ_* and μ^* are measures on $(\Omega_0, \Gamma \restriction_{\Omega_0})$; we call them **inner** and **outer restrictions** of μ to Ω_0 respectively.*
2. *For all $E \in \Gamma$ we have that $\mu_*(E \cap \Omega_0) \leq \mu^*(E \cap \Omega_0) \leq \mu(E)$. If $\mu(\Omega) = \mu^*(\Omega_0) < \infty$, then $\mu^*(E \cap \Omega_0) = \mu(E)$ for all $E \in \Gamma$.*
3. *μ_* and μ^* are measures on $(\Omega, \sigma(\{\Omega_0\} \cup \Gamma))$ which agree with μ on Γ; we call them **inner** and **outer extensions** of μ to $\sigma(\{\Omega_0\} \cup \Gamma)$ respectively.*

Proof See Exercises 1.33 - 1.35 in [HJ94]. ■

It is important to note that, in general, μ_* and μ^* do not give the same extension of μ to $(\Omega, \sigma(\{\Omega_0\} \cup \Gamma))$. For example, if λ is the Lebesgue measure on $[0,1]$ and $N \subseteq [0,1]$ is a Lebesgue non-measurable subset, i.e. $N \notin \mathcal{L}(\lambda)$, then by part (2) of Proposition 1.17 we must have $\lambda_*(N) < \lambda^*(N)$. In fact, it is possible to show that for all $t \in [0,1]$ the function $\lambda_t := t\lambda_* + (1-t)\lambda^*$ is a measure on $([0,1], \sigma(\{N\} \cup \mathcal{L}(\lambda))$ which extends λ [HJ94, Exercise 1.34-(iii)].

Suppose that (Ω, Γ, μ) is a measure space and $f : \Omega \to \overline{\mathbb{R}}$ is an integrable function. The **Lebesgue integral** of f with respect to μ is denoted as: $\int f d\mu$, $\int f(x)\mu(dx)$ or $\mu(f)$.

1.8 Product σ-algebras, measures and Kolmogorov's Theorem

Let $(\Omega_t, \Gamma_t)_{t \in T}$ be a family of measurable spaces where T is an arbitrary set of indices. The **product σ-algebra** $\otimes_t \Gamma_t$ on $\Pi_t \Omega_t$ is defined [Bil86, Chapter 7] as $\sigma(\pi_t : t \in T)$ where $\pi_t : \Pi_t \Omega_t \to \Omega_t$ is the natural projection. Hence, a π-system generating $\otimes_t \Gamma_t$ is given by the set of cylinders:

$$\{\langle E_1, E_2, \ldots, E_n\rangle \mid \{t_1, t_2, \ldots, t_n\} \subseteq_f T\ , E_k \in \Gamma_{t_k} \text{ for } k = 1, \ldots, n\}$$

where

$$\begin{aligned}\langle E_1, E_2, \ldots, E_n\rangle &= \pi_{t_1}^{-1} E_1 \cap \cdots \cap \pi_{t_n}^{-1} E_n \\ &= \{(\omega_t)_{t\in T} \in \Pi_t \Omega_t \mid \omega_{t_k} \in E_k \text{ for } k = 1, \ldots, n\}.\end{aligned}$$

We remind the reader that the symbol $\otimes$ is also used (with a different meaning) to denote the Cartesian product topology. The relationship between the product topology and the product of the Borel σ-algebras of the spaces in question deserves a note of warning.

Remark 1.19 *If $(X_t, \mathcal{G}_t)_{t\in T}$ is a family of topological spaces, then it is not difficult to see that $\otimes_t \mathcal{B}X_t \subseteq \mathcal{B}(\Pi_t X_t) = \sigma(\otimes_t \mathcal{G}_t)$. But in general $\mathcal{B}(\Pi_t X_t)$ is larger than $\otimes_t \mathcal{B}X_t$. If $T = \{t_1, t_2, \ldots, t_n\}$ is finite and $(X_t, \mathcal{G}_t)$ is second countable for all $t = 1,\ldots n$, then $\mathcal{B}X_{t_1} \otimes \cdots \otimes \mathcal{B}X_{t_n} = \mathcal{B}(X_{t_1} \times \cdots \times X_{t_n})$ [Dud89, Proposition 4.1.7].*

Let $(\Omega_t, \Gamma_t)_{t\in T}$ be a family of measurable spaces where T is any set. A family of measures $\{\mu_A \mid A \subseteq_f T\}$ where each μ_A is defined on $(\Pi_{t\in A}\Omega_t, \otimes_{t\in A}\Gamma_t)$ is said to be **consistent** if for all $A, B \in 2_f^T$ with $A \subseteq B$ we have that $\mu_A = \mu_B \circ \pi_{BA}^{-1}$. That is, $\mu_A(E) = \mu_B(\pi_{BA}^{-1}E)$ for all $E \in \otimes_{t\in A}\Gamma_t$. Suppose that ν is a measure on $(\Pi_t X_t, \otimes_t \Gamma_t)$. For every $A \subseteq_f T$ we define the A-**marginal distribution** of ν as $\pi_A[\nu] := \nu \circ \pi_{TA}^{-1}$. It is not difficult to see that $\{\pi_A[\nu] \mid A \subseteq_f T\}$ is a consistent family of measures. The question as to whether given a consistent family of measures there exists a measure in the product space with such a family as marginal distributions is answered by the next proposition.

Proposition 1.20 ***Kolmogorov's Existence Theorem***

Let T be any set and $(X_t, \mathcal{G}_t)_{t\in T}$ be a family of Hausdorff spaces. Suppose that $\mathbb{P}_A$ is a inner regular probability measure defined on $(\Pi_{t\in A}X_t, \otimes_{t\in A}\mathcal{B}X_t)$ for all $A \subseteq_f T$. If the family $\{\mathbb{P}_A\}_{A\subseteq_f T}$ is consistent, then there exists a probability measure $\mathbb{P}_T$ on $(\Pi_t X_t, \otimes_t \mathcal{B}X_t)$ such that $\mathbb{P}_A = \mathbb{P}_T \circ \pi_{TA}^{-1}$ for all $A \subseteq_f T$.

Proof See Theorem 12.1.2 and Exercise 1 (p. 353) in [Dud89]. ■

The original result by Kolmogorov was stated for $X_t = \mathbb{R}$ for all t. In fact the theorem still holds if we replace $\mathcal{B}X_t$ with any σ-algebra Γ_t such that $\mathcal{G}_t \subseteq \Gamma_t$ and $\mathbb{P}_A$ is inner regular on $(\Pi_t X_t, \otimes_t \Gamma_t)$ for all $A \subseteq_f T$. Proposition 4.6 gives a further generalisation.

Suppose that $(\Omega_k, \Gamma_k, \mu_k)$ is a σ-finite measure space for $k = 1,..., n$. Then there exists a unique measure $\Pi_{k=1}^n \mu_k$ on $(\Pi_{k=1}^n X_t, \otimes_{k=1}^n \Gamma_t)$ called the **product measure** such that

$$(\Pi_{k=1}^n \mu_k)(E_1 \times E_2 \times \cdots \times E_n) = \mu_1(E_1) \cdots \mu_n(E_n)$$

for all $E_k \in \Gamma_k$ and $k = 1,..., n$.

Corollary 1.21 *Let T be any set and $(X_t, \mathcal{G}_t)_{t \in T}$ be a family of Hausdorff spaces. Suppose that $\mathbb{P}_t$ is a inner regular probability measure on $(X_t, \mathcal{B}X_t)$ for all $t \in T$. Then there exists a probability measure $\Pi_t \mathbb{P}_t$ on $(\Pi_t X_t, \otimes_t \mathcal{B}X_t)$ such that*

$$\prod\nolimits_t \mathbb{P}_t(\langle E_1, E_2,..., E_n \rangle) = \mathbb{P}_{t_1}(E_1) \cdots \mathbb{P}_{t_n}(E_n)$$

for all $\{t_1, t_2,..., t_n\} \subseteq_f T$ and $E_k \in \mathcal{B}X_{t_k}$ for $k = 1,..., n$

Proof We only need to observe that the finite product of inner regular measures is also inner regular [Mar53, Section 6]. ■

1.9 Probability theory

The references for this section are the same as for Section 1.7. For the rest of the section $(\Omega, \Gamma, \mathbb{P})$ denotes a probability space. Elements of Γ are called events. A property p is said to hold **almost everywhere** (a.e.), if there exists a set $B \in \Gamma$ such that $\{\omega \in \Omega \mid p(\omega) \text{ is false}\} \subseteq B$ and $\mathbb{P}(B) = 0$. If the space $(\Omega, \Gamma, \mathbb{P})$ is complete, then p holds almost everywhere if and only if $\mathbb{P}(\{\omega \in \Omega \mid p(\omega) \text{ is true}\}) = 1$. A **random function** on Ω with values on a measurable space (S, Σ) is a measurable function $x : \Omega \to S$. Recall that $\sigma(x) \subseteq \Gamma$ is the smallest σ-algebra on Ω that makes x measurable; that is, $\sigma(x) = \{x^{-1}B \mid B \in \Sigma\}$. The **distribution** or **law** of x is the probability measure $\mathbb{P}_x$ on (S, Σ) defined as $\mathbb{P}_x(A) := \mathbb{P}(\{\omega \in \Omega \mid x(\omega) \in A\}) = \mathbb{P}(x^{-1}A)$ for all $A \in \Sigma$. The number $\mathbb{P}(x^{-1}A)$ is also denoted as $\mathbb{P}(x \in A)$. A random function x taking values on $(\overline{\mathbb{R}}, \mathcal{B}\overline{\mathbb{R}})$ is called a **random variable**. The **distribution function** $F_x : \mathbb{R} \to [0, 1]$ of a random variable x is defined as $F_x(v) := P(x \leq v)$. A k-dimensional **random vector** is a function $x := (x_1, x_2,..., x_k) : \Omega \to \mathbb{R}^k$ such that x_j is a random variable for $j = 1,..., k$. The law and the distribution function of a k-dimensional random vector $x = (x_1, x_2,..., x_k)$ are defined as $\mathbb{P}_x(A) := \mathbb{P}(x^{-1}A)$ for all $A \in \mathcal{B}\mathbb{R}^k$, and $F_x(v_1, v_2,..., v_k) := \mathbb{P}(\{\omega \mid x_j(\omega) \leq v_j \text{ for } j = 1,..., k\})$ for all $(v_1, v_2,..., v_k) \in \mathbb{R}^k$. Often $\mathbb{P}_x$ and F_x are called **joint** law and joint distribution of the variables $x_1, x_2,..., x_k$.

Let T be any set. A family $\{E_t\}_{t\in T}$ of events is said to be **independent** if for all $A \subseteq_f T$ we have that $\mathbb{P}(\bigcap_{t\in A} E_t) = \Pi_{t\in A}\mathbb{P}(E_t)$. A family $\{\Gamma_t\}_{t\in T}$ of sub-σ-algebras of Γ is independent if for any choice of $E_t \in \Gamma_t$ we have that $\{E_t\}_{t\in T}$ is independent. A family of random functions $\{x_t\}_{t\in T}$, where each x_t takes values on a measurable space (S_t, Σ_t), is independent if $\{\sigma(x_t)\}_{t\in T}$ is independent; that is

$$\mathbb{P}(x_t \in E_t \text{ for all } t \in A) = \textstyle\prod_{t\in A} \mathbb{P}(x_t \in E_t)$$

for all $A \subseteq_f T$ and any choice of $E_t \in \Sigma_t$. The **expectation** $\mathbb{E}(x)$ of an integrable random variable $x : \Omega \to \overline{\mathbb{R}}$ is defined as the Lebesgue integral $\int_\Omega x d\mathbb{P}$. If $\{x_t\}_{t\in T}$ is a family of independent and integrable random variables, then for all $A \subseteq_f T$ the random variable $\Pi_{t\in A} x_t$ is integrable and $\mathbb{E}(\Pi_{t\in A} x_t) = \Pi_{t\in A}\mathbb{E}(x_t)$.

Proposition 1.22 *(Borel-Cantelli Lemmas) Let $(\Omega, \Gamma, \mathbb{P})$ be a probability space and (E_n) be a sequence of events. Then*

1. *If $\sum_{n\in\mathbb{N}} \mathbb{P}(E_n) < \infty$, then $\mathbb{P}(\limsup_n E_n) = 0$.*
2. *If $\sum_{n\in\mathbb{N}} \mathbb{P}(E_n) = \infty$ and (E_n) is independent, then $\mathbb{P}(\limsup_n E_n) = 1$.*

Proof See [HJ94, Section 2.11]. ■

Suppose that $(\Omega, \Gamma, \mathbb{P})$ is a probability space and $(S_t, \Sigma_t)_{t\in T}$ is a family of measurable spaces indexed by an arbitrary set T. A family $x_t : \Omega \to S_t$ of random functions defined for all $t \in T$ is called a **stochastic process** on $(\Omega, \Gamma, \mathbb{P})$ with parameter set T. Notice that $x : \Omega \to \Pi_t S_t$ given by $x(\omega) = (x_t(\omega))_{t\in T}$ is a random function taking values on $(\Pi_t S_t, \otimes_t \Sigma_t)$. Hence $\mathbb{P}_x(E) = x^{-1}E$ for all $E \in \otimes_t \Sigma_t$ is a probability measure on $(\Pi_t S_t, \otimes_t \Sigma_t)$. For $A = \{t_1, t_2, \ldots, t_n\} \subseteq_f T$ we define the A-**finite dimensional distribution** of $\{x_t\}_{t\in T}$ as:

$$\begin{aligned}
\mathbb{P}_{t_1,\ldots,t_n}(E) &= \pi_A[\mathbb{P}_x](E) \\
&= \mathbb{P}_x(\pi_{TA}^{-1} E) \\
&= \mathbb{P}_x(\{\omega \mid (x_{t_1}(\omega), \ldots, x_{t_n}(\omega)) \in E\})
\end{aligned}$$

for all $E \in \otimes_{t\in A}\Sigma_t$. In particular $\mathbb{P}_t := \pi_{\{t\}}[\mathbb{P}_x] = \mathbb{P}_{x_t}(E) = \mathbb{P}(x_t^{-1}E)$ for all $t \in T$. Clearly $\{\mathbb{P}_{t_1,\ldots,t_n} \mid \{t_1, \ldots, t_n\} \subseteq_f T \text{ and } n \in \mathbb{N}\}$ is a consistent family of probability measures. If $\{x_t\}_{t\in T}$ is independent, then $\mathbb{P}_x = \Pi_t \mathbb{P}_t$. Conversely if we ask ourselves whether there exists a stochastic process with a given set of finite-dimensional distributions, the answer is given by Kolmogorov's Theorem (Proposition 1.20).

1.10 Valuations

For an in-depth treatment of spaces of valuations the reader is referred to [Hec96]. Given a topological space $(X, \mathcal{G})$, we say that $\nu : \mathcal{G} \to \overline{\mathbb{R}}_+$ is a **valuation** on X [Law82][JP89][Tix95][Kir93] if it satisfies:

- $\nu(\emptyset) = 0$ (strictness)
- $G_1 \subseteq G_2 \Rightarrow \nu(G_1) \leq \nu(G_2)$ (monotonicity)
- $\nu(G_1 \cup G_2) + \nu(G_1 \cap G_2) = \nu(G_1) + \nu(G_2)$ (modularity).

A **valuation space** is a triple $(X, \mathcal{G}, \nu)$ where $(X, \mathcal{G})$ is a topological space and ν is a valuation on X. Let $(X, \mathcal{G}, \nu)$ be a valuation space, then

- ν is **finite** if $\nu(X) < \infty$.
- ν is **normalised** if $\nu(X) = 1$.
- ν is **σ-finite** if there exists a sequence of open sets $(G_n)_{n \in \mathbb{N}}$ such that $X = \bigcup_{n \in \mathbb{N}} G_n$ and $\nu(G_n) < \infty$ for all $n \in \mathbb{N}$.
- ν is **locally finite** if for all $x \in X$ there exists $G \in \mathcal{G}$ such that $x \in G$ and $\nu(G) < \infty$.
- ν is (Scott) **continuous** if for every directed family $\mathcal{G}^* \subseteq \mathcal{G}$ we have that $\nu(\bigcup^{\uparrow} \mathcal{G}^*) = \sup_{G \in \mathcal{G}^*} \nu(G)$; in other words ν is a Scott continuous map on $(\mathcal{G}, \subseteq)$. Notice that any τ-smooth measure is a continuous valuation when restricted to the open sets.
- ν is **co-continuous** if for every filtered family $\mathcal{G}^* \subseteq \mathcal{G}$ we have that $\nu(int(\bigcap_{\downarrow} \mathcal{G}^*)) = \inf_{G \in \mathcal{G}^*} \nu(G)$.
- ν is **bi-continuous** if it is continuous and co-continuous.

Clearly, ν finite $\Rightarrow$ ν σ-finite $\Rightarrow$ ν locally finite. For any $a \in X$ we define the **point valuation** based at a as the function $\delta_a : \mathcal{G} \to \{0, 1\}$ such that $\delta_a(G) = 1$ if $a \in G$. A **simple valuation** ν is any finite linear combination $\sum_{i=1}^{n} r_i \delta_{a_i}$ of point valuations with $r_i \in \mathbb{R}_+ \backslash \{0\}$ for $i = 1,...,n$. The set $\{a_1,...,a_n\}$ is called the support of ν and is denoted by $|\nu|$. Simple valuations are always continuous. A simple valuation can be extended to a measure on the whole power set of X and its extension to a measure on $\mathcal{B}X$ is unique [Kir93][Tix95]. We use $\mathbf{V}X$ to denote the set of all continuous valuations on X. For $\mu, \nu \in \mathbf{V}X$ we define: $\mu \sqsubseteq \nu$ if and only if $\mu(G) \leq \nu(G)$ for all $G \in \mathcal{G}$. $(\mathbf{V}X, \sqsubseteq)$ is a dcpo having the constant zero valuation as least element. For any directed subset $\{\nu_i\}_{i \in I}$ of $\mathbf{V}X$, we have that $(\bigsqcup^{\uparrow}_{i \in I} \nu_i)(G) = \sup_{i \in I} \nu_i(G)$

for all $G \in \mathcal{G}$. The **probabilistic powerdomain** [SD80][Gra88][JP89] of X is defined as $\mathbf{P}X := \{\nu \in \mathbf{V}X \mid \nu(X) \leq 1\}$. The **normalised** probabilistic powerdomain of X is defined as $\mathbf{P}^1 X := \{\nu \in \mathbf{V}X \mid \nu(X) = 1\}$. If D is a dcpo with least element $\bot$, then $\delta_\bot$ is the least element of $\mathbf{P}^1 D$. Suppose that $(X, \mathcal{G}, \nu)$ is a valuation space and $H \in \mathcal{G}$, then the **restriction** ν_H, defined as $\nu_H := \nu(G \cap H)$ for all $G \in \mathcal{G}$, is clearly a valuation on X. Furthermore, if ν is continuous, then ν_H is continuous for all $H \in \mathcal{G}$ [Hec96, Section 3.3]. In Chapter 6 (see remarks to Corollary 6.16) we discuss the proper way to topologise $\mathbf{V}X$.

Co-continuous valuations were introduced by Bukatin and Shorina in [BS98] (see also [Kei98]); their definition is provided for completeness but we will not make any further use of them. Also for completeness, we mention the work of O'Neill [O'N96] and Heckmann [Hec99] on partial metrics on spaces of valuations. O'Neill's definition of valuation is more general than ours, since it is stated for a certain class of semilattices as opposed to the lattice of open sets of a topological space.

1.11 Basic extension results

The following results are the core of our standard machinery for extending valuations to Borel measures.

Proposition 1.23

1. *(Smiley-Horn-Tarski) Let $(X, \mathcal{G})$ be a topological space. A finite valuation $\nu : \mathcal{G} \to \mathbb{R}_+$ extends uniquely to a content $\overline{\nu} : \mathcal{A}(\mathcal{G}) \to \mathbb{R}_+$.*
2. *Let $\mathcal{S}$ be a semialgebra and let $\mu : \mathcal{S} \to \overline{\mathbb{R}}_+$ be a set function. Suppose that μ is finitely additive and σ-subadditive on $\mathcal{S}$, then μ can be extended uniquely to a measure on $\sigma(\mathcal{S})$.*
3. *Suppose that $\mathcal{I}$ is a π-system on X and μ, ν are finite measures on $(X, \sigma(\mathcal{I}))$. If $\mu(E) = \nu(E)$ for all $E \in \mathcal{I}$, then $\mu = \nu$.*

Proof

1. See Theorems 1.9 and 1.22 in [HT48], [Smi44], Theorem 1.2 in [Pet51] or corollary to Lemma 8.1 in [Top78].
2. See Theorem 11.3 in [Bil86].
3. See Lemma 1.6-(b) in [Wil91]. ■

Remark 1.24 *Part (1) of the previous proposition is commonly known as* *the Smiley-Horn-Tarski extension theorem; but according to Z. Lipecki this result goes back to F. Hausdorff [Hau49].*

Notice that by part (3) of the last proposition if two finite Borel measures agree on the open sets then they are equal. Hence the extension of a finite valuation to a Borel measure is unique when it exists. Let $(X, \mathcal{G})$ be a topological space. We define the paving of **crescents**, or semiopen subsets of X, as

$$\begin{aligned} Cres(X) &:= \{G_1 \backslash G_2 \mid G_1, G_2 \in \mathcal{G}\} \\ &= \{G \cap F \mid G \in \mathcal{G},\, F \in \mathcal{F}\} \end{aligned}$$

It is easy to verify that $Cres(X)$ is a semialgebra on X and that $\mathcal{G} \cup \mathcal{F} \subseteq Cres(X)$. Notice that $\mathcal{B}X = \sigma(Cres(X))$. As an important corollary to Proposition 1.23 we have,

Corollary 1.25 ***Basic Extension of Valuations***

Let $(X, \mathcal{G})$ be a topological space and $\nu : \mathcal{G} \to \mathbb{R}_+$ be a finite valuation. If the extension $\overline{\nu} : Cres(X) \to \mathbb{R}_+$ given by $\overline{\nu}(G_1 \backslash G_2) := \nu(G_1) - \nu(G_1 \cap G_2)$ is σ-subadditive, then ν extends uniquely to a Borel measure on $(X, \mathcal{B}X)$.

Proof By part (1) of Proposition 1.23, ν extends uniquely to a content $\overline{\nu}$ on $\mathcal{A}(\mathcal{G})$. Since $Cres(X) \subseteq \mathcal{A}(\mathcal{G})$ we have that $\overline{\nu}(G_1 \backslash G_2) = \overline{\nu}(G_1) - \overline{\nu}(G_2)$ for all G_1, G_2 in $\mathcal{G}$ such that $G_2 \subseteq G_1$. If we assume that $\overline{\nu}$ is σ-subadditive on $Cres(X)$ then by part (2) it can be extended uniquely to a measure on $(X, \mathcal{B}X)$. ■

1.12 Metric, Polish and analytic spaces

A **metric** or distance function d on a set X is a map $d : X \times X \to \mathbb{R}_+$ satisfying for all $x, y, z \in X$:

1. $d(x, y) = 0$ if and only if $x = y$.
2. $d(x, y) = d(y, x)$ (symmetry).
3. $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality).

A metric d on X which satisfies $d(x, z) \leq \max(d(x, y), d(y, z))$ for all $x, y, z \in X$ is called an **ultrametric**. If d is a metric on X we say that (X, d) is a **metric space**. For the rest of this section (X, d) denotes a metric space. The set $B(x, r) := \{y \in X \mid d(x, y) < r\}$ is called the open d-**ball** of radius r and center x. Notice that $r_1 < r_2$ implies that $B(x, r_1) \subseteq B(x, r_2)$ and that $B(x, 0) = \emptyset$. The set $\{B(x, r) \mid x \in X \text{ and } r > 0\}$ is a base for a topology

$\mathcal{T}(d)$ called the topology **induced** in X by the metric d. When we refer to a topological property of a metric space (X, d) it is assumed to be in relation to $\mathcal{T}(d)$. A topological space $(Y, \mathcal{G})$ is called **metrizable** if there exists a metric d on Y such that $\mathcal{G} = \mathcal{T}(d)$. A sequence (x_n) in X is called a **Cauchy** sequence if for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $m, n \geq N$ implies that $d(x_m, x_n) < \varepsilon$. Notice that all convergent sequences are Cauchy. We say that (X, d) is a **complete metric space** or that the metric d is **complete** on X if every Cauchy sequence in X converges. Every metric on a compact metric space is complete [Dug66, Corollary XIV-2.4]. For any metric space (X, d), there exists a complete metric space $(\overline{X}, \overline{d})$ such that X is a dense subspace of $\overline{X}$ and $\overline{d}$ extends d. The space $(\overline{X}, \overline{d})$ is unique up to an isometry and it is called the **completion** of (X, d) [Dug66, Section XIV-6]. A space $(Y, \mathcal{G})$ is (topologically) **complete** if it is metrizable with a complete metric. A space which is metrizable with a complete and separable metric is called a **Polish** space. Let $I^\infty := \Pi_{n\in\mathbb{N}}[0, 1]$ with the Cartesian product topology. Then I^∞ is a compact Polish space called the **Hilbert cube** (see Section IX-8 in [Dug66] for a more complete definition). The following results give us some insight into the nature of metric spaces and metrization.

Proposition 1.26 *Let $(X, \mathcal{G})$ be a topological space.*

1. *Every metric space is T_4 (Hausdorff normal) and first countable.*
2. *In metric spaces second countability, separability and fully Lindelöf are equivalent properties.*
3. *(Urysohn) Suppose that X is second countable. Then X is metrizable if and only if it is T_3. In that case X is homeomorphic to a subset of the Hilbert cube I^∞.*
4. *(Mazurkiewicz) Suppose that X is complete. Then $B \subseteq X$ is complete if and only if B is a G_δ subset of X.*
5. *Every locally compact Hausdorff space is completely regular (hence T_3).*
6. *Every locally compact metric space is topologically complete.*

Proof The proof for these statements can be found in [Dug66] as follows: part (1) Theorems VIII-2.2, IX-5.4 and IX-5.5; part (2) Theorem IX-5.6; part (3) Corollary IX-9.2; part (4) Theorem XIV-8.3; part (5) Theorem XI-6.4; part (6) Corollary XIV-2.4. ■

As an important corollary we have

Corollary 1.27

1. *Every second countable locally compact Hausdorff space is Polish.*
2. *Every Polish space is homeomorphic to a G_δ subset of the Hilbert cube I^∞.*

Proof Let $(X, \mathcal{G})$ be a topological space.

1. Suppose that X is second countable, locally compact and Hausdorff. By part (5) X is completely regular, hence regular. By part (3) there exists a metric d_1 such that $\mathcal{T}(d_1) = \mathcal{G}$, hence (X, d_1) is a locally compact metric space. By part (6) we can find a complete metric d_2 such that $\mathcal{T}(d_2) = \mathcal{G}$. By part (2) we conclude that (X, d_2) is separable and hence $(X, \mathcal{G})$ is Polish.
2. Suppose that X is Polish. Let d be a metric such that (X, d) is a complete separable metric space. By part (2) we have that X is second countable. By part (3) X is a complete subspace of I^∞. Since I^∞ is complete the conclusion follows from part (4). ■

Let $(X, \mathcal{G}_1)$, $(Y, \mathcal{G}_2)$ be Polish spaces and $B \subseteq Y$ be a nonempty subset. Then the following conditions are equivalent [Dud89, Theorem 13.2.1]:

- $B = fX$ for a continuous function $f : X \to Y$.
- $B = fX$ for a Borel measurable function $f : X \to Y$.
- $B = fA$ for some Borel subset $A \subseteq X$ and $f : A \to Y$ continuous.
- $B = fA$ for some Borel subset $A \subseteq X$ and $f : A \to Y$ Borel measurable.

A set B satisfying the previous conditions is called an **analytic set**. The set B with the subspace topology inherited from Y is called an **analytic space**. Notice that any Borel subset of a Polish space is an analytic set. Any analytic set whose complement is analytic is a Borel set. Analytic sets are separable and closed under countable unions and intersections. The image of an analytic space under a Borel measurable function into a Polish space is analytic. Therefore the image of a analytic space under a Borel map into an analytic space is analytic. It is important to remember that analytic subsets of a Hausdorff space are universally measurable [Dud89, Theorem 13.2.6][THJ80, Equations (34) and (35)] but there exist analytic subsets of a Polish space which are not Borel measurable [Dud89, Proposition 13.2.5]. The reader is referred to [Dud89, Section 13.2] for more information; an in-depth treatment of analytic sets can be found in [RJD$^+$80]. The following results are important.

Proposition 1.28 *Let $(X, \mathcal{G})$ be a Hausdorff space and μ be a Borel measure on X.*

1. *If X is analytic and μ is σ-finite, then μ is inner regular. In particular, $\mathfrak{M}^1(X) = \mathfrak{M}^1(X; F) = \mathfrak{M}^1(X; R) = \mathfrak{M}^1(X; \tau)$.*

2. *If X is locally compact, every set in $\mathcal{G}$ is σ-compact and μ is finite on $\mathcal{K}X$, then μ is regular (inner regular and outer regular).*

3. *Suppose that X is topologically complete or locally compact. If μ is finite and τ-smooth, then μ is inner regular. In particular, $\mathfrak{M}^1(X; R) = \mathfrak{M}^1(X; \tau)$.*

Proof (1) See (P.5.19) in [HJ94] or (A4) in [THJ80]. (2) See Theorem 2.18 in [Rud87]. (3) See (P16) in [Top70b, page XIII]. ■

Proposition 1.29 *Let $(X, \mathcal{G})$ be a Hausdorff space and μ be a Borel measure on X.*

1. *If μ is inner regular, then μ is $\mathcal{F}$-regular. That is, $\mathfrak{M}(X; R) \subseteq \mathfrak{M}(X; F)$.*

2. *If every closed subset of X is G_δ (e.g. if X is metrizable) and μ is finite, then μ is $\mathcal{F}$-regular.*

3. *If X is regular and μ is finite and τ-smooth, then μ is $\mathcal{F}$-regular. In particular, $\mathfrak{M}^1(X; \tau) \subseteq \mathfrak{M}^1(X; F)$.*

4. *If X is metrizable, μ is finite and there exists a separable Borel set $Y \in \mathcal{B}X$ with $\mu(X \backslash Y) = 0$, then μ is τ-smooth.*

5. *If X is metrizable, then the assertion "all finite Borel measures on X are τ-smooth" is independent of the axioms of set theory.*

6. *If X is compact, then for finite measures: regularity, $\mathcal{F}$-regularity, inner regularity and τ-smoothness are equivalent concepts. In particular, $\mathfrak{M}^1(X; F) = \mathfrak{M}^1(X; R) = \mathfrak{M}^1(X; \tau)$.*

7. *If X is metrizable and compact, then all finite measures are Radon, τ-smooth and $\mathcal{F}$-regular. In particular, $\mathfrak{M}^1(X) = \mathfrak{M}^1(X; F) = \mathfrak{M}^1(X; R) = \mathfrak{M}^1(X; \tau)$.*

Proof Part (1) is obvious since in Hausdorff spaces $\mathcal{K} \subseteq \mathcal{F}$. Parts (2) and (3) appear without proof in [Top70b, (P16), page XIII] but are corollaries to the very general Theorems 2 and 3 in [Top70a]. For parts (4) and (5) see

Section 1.34 and Exercise 1.61 in [HJ94]; part (4) also follows from part (2) of Proposition 1.26 which implies that Y with the subspace topology is second countable and we noticed that in fully Lindelöf spaces Borel measures are τ-smooth. Part (6) follows immediately from part (3) of Proposition 1.28 and the fact that in compact Hausdorff spaces $\mathcal{K} = \mathcal{F}$ [Dug66, Theorem XI-(1.4)-3]. Part (7) is immediate from part (2). ■

2. Extension of valuations on dcpos

2.1 Summary and historic note

In this chapter we give an extension result for continuous valuations on dcpo's. Namely, we show that if a locally finite continuous valuation is the supremum of a directed family of simple valuations, then it can be extended uniquely to a τ-smooth Borel measure (Theorem 2.21). This result is not only stated for dcpo's endowed with the Scott topology, but rather for all monotone convergence spaces. The most important corollary (Corollary 2.22) is, of course, that every locally finite continuous valuation on a continuous dcpo extends uniquely to a Borel measure. This last result will also be derived later, using a totally different technique, as a consequence of the Extension Theorem for Locally Compact Sober Spaces (Theorem 3.27). We show that if a topological space is such that every finite continuous valuation extends uniquely to a measure, then this is also the case for every locally finite continuous valuation (Theorem 2.13). In fact, the moral of Section 2.3 seems to be that everything which is true for finite continuous valuations holds true for locally finite continuous valuations. Finally we show that there exists a finite continuous valuations on a dcpo (endowed with the Scott topology) that does not extend to a Borel measure. In fact, we show that this is the case for all T_0 and non-sober spaces whose sobrification is first countable (Theorem 2.34).

As we mentioned in the introduction, the extension result for ω-algebraic dcpo's in [SD80] contained a gap; nevertheless, the original idea of Saheb-Djahromi (Lemma 2 in [SD80]) could still be recovered and used for obtaining the extension theorem for dcpo's. The key result to establish was Lemma 2.14. This Lemma is proved in two different ways; both of them rely heavily on a choice argument. The proof in Section 2.6 uses only partial order theory. The proof in Section 2.5 was developed jointly with A. Edalat and N. Saheb-Djahromi, and corresponds to Proposition 3.2 in [AMESD00]. The choice argument is hidden in the properties of compact sets and compact measures as Tychonoff's and Kolmogorov's theorems. Theorem 2.16, Corollaries 2.20 and 2.24, and some auxiliary results also appear in [AMESD00].

It is important to notice that Theorem 2.21 and Corollary 2.22 generalise

Theorem 5.1 and Corollary 4.3 in [AMESD00] which were only given for σ-finite and finite valuations. This is done through Lemma 2.11. The careful reader will notice that Theorem 2.21 does not follow directly from Theorems 2.13 and 2.16, which explains the existence of Lemma 2.11 in its rather convoluted form.

Finally, we note that the example in Section 2.7 also appears in [AMESD00]. By abstracting the situation in this example we have been able to obtain the general negative result, namely Theorem 2.34.

2.2 Auxiliary results

Recall that any simple valuation $\nu := \sum_{x\in|\nu|} r_x\delta_x$ on a space $(X,\mathcal{G})$ can be extended to a measure $\overline{\nu} : 2^X \to \mathbb{R}_+$ given by $\overline{\nu}(B) := \sum_{x\in|\nu|\cap B} r_x$ for all $B \subseteq X$. If $(D_1,\sqsubseteq_1)$ and $(D_2,\sqsubseteq_2)$ are dcpo's, then the Cartesian product $(D_1 \times D_2,\sqsubseteq)$ with $(x_1,x_2) \sqsubseteq (y_1,y_2)$ if $x_i \sqsubseteq_i y_i$ for $i = 1,2$, is a dcpo.

Proposition 2.1 *Suppose that D_1, D_2 and D are dcpo's. A function $f : D_1 \times D_2 \to D$ is Scott continuous if and only if it is continuous in each variable separately. That is, $f(\_,y) : D_1 \to D$ is Scott continuous for every $y \in D_2$ and similarly for D_1.*

Proof See Lemma 3.2.6 in [AJ94]. ■

Lemma 2.2 *Consider the continuous dcpo $(\overline{\mathbb{R}},\leq)$, then $+ : \overline{\mathbb{R}} \times \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ is a Scott continuous function.*

Proof Let $\langle x_j\rangle_{j\in J}$ be a directed subset of $\overline{\mathbb{R}}$. Then $\bigsqcup_j^{\uparrow} x_j + y = \sup_j x_j + y = \sup_j (x_j + y)$. In the same way we have that $+$ is continuous on the second argument, therefore it is Scott continuous. ■

Lemma 2.3 *Let $(X,\mathcal{G})$ be a topological space and ν be a continuous valuation on X. Then ν is locally finite if and only if there exists a directed family of open sets $\{G_j\}_{j\in I}$ such that $X = \bigcup_{j\in I}^{\uparrow} G_j$ and $\nu(G_j) < \infty$ for all $j \in I$.*

Proof $\boxed{\Leftarrow}$ It is obvious. $\boxed{\Rightarrow}$ If ν is locally finite, then for all $x \in X$ there exist $G_x \in \mathcal{G}$ such that $x \in G_x$ and $\nu(G_x) < \infty$. Then by modularity of ν for all finite $F \subseteq_f X$ we have that $\nu(\bigcup_{x\in F} G_x) \leq \sum_{x\in F} \nu(G_x) < \infty$. Therefore the set $\{\bigcup_{x\in F} G_x \mid F \subseteq_f X\}$ is directed and satisfies the hypothesis. ■

The following lemma will be discussed extensively and generalised in Section 7.5.

Lemma 2.4 ***Splitting Lemma*** *(normalised version)*

Let $(D, \mathcal{G})$ *be a monotone convergence space and* $\nu_1 := \sum_{b\in|\nu_1|} r_b\delta_b$, $\nu_2 := \sum_{c\in|\nu_2|} s_c\delta_c \in \mathbf{P}^1 D$. *Then* $\nu_1 \sqsubseteq \nu_2$ *if and only if there exists a function* $f : |\nu_1| \times |\nu_2| \to [0,1]$ *such that for all* $b \in |\nu_1|$ *and* $c \in |\nu_2|$

1. $\sum_{c\in|\nu_2|} f(b,c) = r_b$.
2. $\sum_{b\in|\nu_1|} f(b,c) = s_c$.
3. $f(b,c) > 0$ *implies* $b \sqsubseteq c$.

Proof See Section 7.5. ■

It is important to notice that for two simple valuations there might exist an uncountable number of splitting maps and there is no canonical way of choosing one. Using the notation of Lemma 2.4, notice that the simple valuation $\lambda := \sum_{(b,c)\in|\nu_1|\times|\nu_2|} f(b,c)\delta_{(b,c)}$ on $D \times D$ extends uniquely to a probability measure $\overline{\lambda}$ on $\mathcal{B}D \otimes \mathcal{B}D$. The natural projections $\pi_1, \pi_2 : D \times D \to D$ are random functions such that $\mathbb{P}_{\pi_i} = \overline{\nu}_i$. Furthermore, if for all $b \in |\nu_1|$ and $c \in |\nu_2|$ we have that $b \sqsubseteq c$, then f can be defined as $f(b,c) := r_b s_c$; it follows that $\overline{\lambda} = \overline{\nu}_1 \times \overline{\nu}_2$ and the random functions π_1 and π_2 are independent. The following result is important.

Proposition 2.5 *Suppose that* D *is a continuous (*ω*-continuous) dcpo. Then*

1. *(Jones and Plotkin)* $\mathbf{P}D$ *is a continuous (*ω*-continuous) dcpo and has a basis of simple valuations.*
2. *(Edalat) If* D *has a least element, then* $\mathbf{P}^1 D$ *is a continuous (*ω*-continuous) dcpo and has a basis of normalised simple valuations.*
3. *(Kirch)* $\mathbf{V}D$ *endowed with the Scott topology is a continuous (*ω*-continuous) dcpo and has a basis of simple valuations.*

Proof (1) See Theorem 5.2 and Corollary 5.4 in [Jon89] (Theorem 8.2 in [JP89]). (2) See Corollary 3.3 in [Eda95b]. (3) See Satz 5.2, 5.3 and Korollar 5.1 in [Kir93]. Heckmann has shown a more general result for point continuous valuations which implies this case (see Lemmas 5.2, 5.4 and Theorem 6.9 in [Hec96]). ■

The following proposition is similar to Theorem 2 in [SD80], and it can be derived from the splitting lemma; here we give a direct proof.

Lemma 2.6 *Let* $(X, \mathcal{G})$ *be any topological space and* ν_1 *and* ν_2 *be simple valuations in* $\mathbf{P}^1 X$. *Then* $\nu_1 \sqsubseteq \nu_2$ *if and only if any of the following two* *conditions hold,*

1. $\overline{\nu}_1(B) \leq \overline{\nu}_2(\uparrow B)$ *for all* $B \subseteq X$*;*
2. $\overline{\nu}_2(B) \leq \overline{\nu}_1(\downarrow B)$ *for all* $B \subseteq X$*.*

Proof Suppose that $\nu_1 \sqsubseteq \nu_2$.

1. Let $E := (\uparrow B)^{\mathsf{c}} \cap (|\nu_1| \cup |\nu_2|)$. Observe that E is a finite set since $E \subseteq |\nu_1| \cup |\nu_2|$. It follows that $\downarrow E$ is closed. Hence $G := (\downarrow E)^{\mathsf{c}}$ is open. We claim that $|\nu_1| \cap G = |\nu_1| \cap \uparrow B$. $\boxed{\subseteq}$ Notice that $|\nu_1| \cap G = |\nu_1| \cap (\downarrow E)^{\mathsf{c}} \subseteq |\nu_1| \cap E^{\mathsf{c}} = |\nu_1| \cap \uparrow B$. $\boxed{\supseteq}$ Since the complement of an upper set is lower we have that $\downarrow(\uparrow B)^{\mathsf{c}} = (\uparrow B)^{\mathsf{c}}$; hence $\downarrow E \subseteq (\uparrow B)^{\mathsf{c}}$. Taking complements in the last inclusion we get $G \supseteq \uparrow B$ and the claim follows.

   By symmetry we have that $|\nu_2| \cap G = |\nu_2| \cap \uparrow B$, hence

$$\begin{aligned}
\overline{\nu}_1(B) &\leq \overline{\nu}_1(\uparrow B) \\
&= \nu_1(G) && \text{since } |\nu_1| \cap G = |\nu_1| \cap \uparrow B \\
&\leq \nu_2(G) && \text{since } \nu_1 \sqsubseteq \nu_2 \text{ and } G \text{ is open} \\
&= \overline{\nu}_2(\uparrow B) && \text{since } |\nu_2| \cap G = |\nu_2| \cap \uparrow B
\end{aligned}$$

2. Let $E := B \cap (|\nu_1| \cup |\nu_2|)$. Again $\downarrow E$ is closed since E is finite and we have that

$$\begin{aligned}
\overline{\nu}_2(B) &= \overline{\nu}_2(E) \\
&\leq \overline{\nu}_2(\downarrow E) = 1 - \overline{\nu}_2(\downarrow E)^{\mathsf{c}} \\
&\leq \overline{\nu}_1(\downarrow E) = 1 - \overline{\nu}_1(\downarrow E)^{\mathsf{c}} && \text{since } \nu_1 \sqsubseteq \nu_2 \text{ and } (\downarrow E)^{\mathsf{c}} \text{ is open} \\
&\leq \overline{\nu}_1(\downarrow B). && \text{since } E \subseteq B.
\end{aligned}$$

The converse implications are trivial since open sets are upper and closed sets are lower. ■

Corollary 2.7 *Let* $(X, \mathcal{G})$ *be any topological space and* ν_1, ν_2 *be simple valuations in* $\mathbf{P}^1 X$*. If* $\nu_1 \sqsubseteq \nu_2$*, then* $|\nu_2| \subseteq \uparrow|\nu_1|$ *and* $|\nu_1| \subseteq \downarrow|\nu_2|$*.*

Proof By the previous lemma $1 = \overline{\nu}_1(|\nu_1|) \leq \overline{\nu}_2(\uparrow|\nu_1|)$ implies that $\overline{\nu}_2(\uparrow|\nu_1|) = 1$. In the same way $1 = \overline{\nu}_2(|\nu_2|) \leq \overline{\nu}_1(\downarrow|\nu_2|)$ implies that $\overline{\nu}_1(\downarrow|\nu_2|) = 1$. ■

The next proposition is a reformulation of Lemma 1 of [SD80] for monotone convergence spaces.

Lemma 2.8 *Let* $(D, \mathcal{G})$ *be a monotone convergence space. If* $T \in Cres(D)$ *then*

1. *If* $x, y \in T$ *and* $x \sqsubseteq_s z \sqsubseteq_s y$ *then* $z \in T$*. In fact, this holds for any*

topological space.

2. *If $S \subseteq T$ is a directed subset then $\bigsqcup^{\uparrow} S \in T$.*

3. *If $S \subseteq D$ is a directed subset and $\bigsqcup^{\uparrow} S \in T$ then there exists $d_0 \in S$ such that $d_0 \in T$.*

Proof Assume that $T = U \backslash V$ with $U, V \in \mathcal{G}$ and $V \subseteq U$. Recall for monotone convergence spaces $\mathcal{G} \subseteq \mathcal{G}_\sigma$.

1. Suppose that $x, y \in T$ and $x \sqsubseteq_s z \sqsubseteq_s y$. We have that $z \in U$ since $x \in U$ and U is upper. For the same reason, if $z \in V$ then we would have $y \in V$ which is false. Therefore $z \notin V$ and we conclude that $z \in T$.

2. Suppose that $S \subseteq T$ is a directed set. Then $\bigsqcup^{\uparrow} S \in U$ since U is upper. If $\bigsqcup^{\uparrow} S \in V$ then, since V is Scott open, there exists $d \in S$ such that $d \in V$. This contradicts $S \subseteq T$. Therefore $\bigsqcup^{\uparrow} S \notin V$ and $\bigsqcup^{\uparrow} S \in T$.

3. Since $\bigsqcup^{\uparrow} S \in U$ and U is Scott open, then there exists $d_0 \in S$ with $d_0 \in U$. If there exists $d \in S \cap V$ then $\bigsqcup^{\uparrow} S \in V$, since V is upper, contradicting $\bigsqcup^{\uparrow} S \in T$. Therefore $S \subseteq V^{\mathsf{c}}$ and $d_0 \in T$. ■

The following result is folklore in domain theory, a proof appears in [Nor89, Proposition 3.1] but one of the steps does not seem to be clear. The argument in the following proof was kindly provided to the author by Professor J. Lawson in an oral discussion.

Lemma 2.9 *Let D be a continuous dcpo. Then D is ω-continuous if and only if $\mathcal{G}_\sigma$ is second countable.*

Proof (Lawson) $\boxed{\Rightarrow}$ Suppose that D is an ω-continuous dcpo with countable basis B. Then it is easy to see that the set $\{\twoheaduparrow b \mid b \in B\}$ is a countable base for the topology. $\boxed{\Leftarrow}$ Suppose that S is a countable base for the Scott topology. Consider the set $P := \{(u, v) \mid u, v \in S \text{ and } \exists w \in D \text{ with } u \subseteq \uparrow w \subseteq v\}$. Using a choice function we can construct a set $B = \{w_p \in D \mid p \in P\}$ such that for all $w_{(u,v)} \in B$ we have that $u \subseteq \uparrow w_{(u,v)} \subseteq v$. Since $P \subseteq S \times S$ then B is a countable set.

We claim that B has the interpolation property (see Proposition 1.3); that is, if $x, z \in D$ and $x \ll z$ then there exists $w \in B$ such that $x \ll w \ll z$. Suppose that $x, z \in D$ with $x \ll z$. Since D is a continuous dcpo then there exists $y \in D$ such that $x \ll y \ll z$. Therefore $y \in \twoheaduparrow x$ and $z \in \twoheaduparrow y$. It follows that there exist two basic open sets $v, u \in S$ such that $y \in v \subseteq \twoheaduparrow x$ and $z \in$

$u \subseteq \Uparrow y$. Hence $u \subseteq \Uparrow y \subseteq \uparrow y \subseteq v$. As a consequence we have that $(u,v) \in P$. Therefore there exists $w_{(u,v)} \in B$ such that $z \in u \subseteq \uparrow w_{(u,v)} \subseteq v \subseteq \Uparrow x$. We conclude that $x \ll w_{(u,v)} \ll z$ with $w_{(u,v)} \in B$. Since D is a continuous dcpo, B is a basis by Proposition 1.3. ■

A valuation ν on a topological space $(X, \mathcal{G})$ is called **countably continuous** if for all ω-chains $\langle G_i \rangle_{i\in\mathbb{N}}$ in $\mathcal{G}$ we have that $\nu(\bigcup_{i\in\mathbb{N}} G_i) = \sup_{i\in\mathbb{N}} \nu(G_i)$.

Lemma 2.10 *Let D be an ω-continuous dcpo. A valuation ν on D is continuous if and only if it is countably continuous.*

Proof $\boxed{\Rightarrow}$ Is immediate. $\boxed{\Leftarrow}$ Let $B := \{b_n \mid n \in \mathbb{N}\}$ be a countable basis for the dcpo. Suppose that we have a directed union $G := \bigcup^{\uparrow}_{k\in K} G_k$ where $G_k \in \mathcal{G}_\sigma$ for all $k \in K$. We must show that $\nu(G) = \sup_{k\in K} \nu(G_k)$.

Since ν is a valuation, by monotonicity we have that $\sup_{k\in K} \nu(G_k) \leq \nu(G)$. Let $c_i := \{b_1, b_2, ..., b_i\}$ for all $i \in \mathbb{N}$. Since B is a basis and G is an open set, for all $x \in G$ there exists $m \in \mathbb{N}$ such that $b_m \ll x$ and $b_m \in G$. Hence $x \in \Uparrow b_m \subseteq \Uparrow(G \cap c_m)$. Therefore $G = \bigcup_{i\in\mathbb{N}} \Uparrow(G \cap c_i)$ and $\{\Uparrow(G \cap c_i)\}_{i\in\mathbb{N}}$ is an ω-chain in ΩD. Since we are assuming that ν is a countably continuous valuation we have that $\nu(G) = \sup_{i\in\mathbb{N}} \nu(\Uparrow(G \cap c_i))$. Observe that for all $i \in \mathbb{N}$ and $b \in G \cap c_i$ there exists a $k \in K$ such that $b \in G_k$. Since the union $\bigcup^{\uparrow}_{k\in K} G_k$ is directed we can find $k_i \in K$ such that $G \cap c_i \subseteq G_{k_i}$. Therefore for all $i \in \mathbb{N}$ we have that $\Uparrow(G \cap c_i) \subseteq G_{k_i}$. Hence $\nu(\Uparrow(G \cap c_i)) \leq \nu(G_{k_i}) \leq \nu(G)$ for all $i \in \mathbb{N}$. We conclude that $\sup_{k\in K} \nu(G_k) = \nu(G)$. ■

In fact, a similar argument shows that the previous result holds for all second countable topological spaces; this fact was brought to the author's attention by Dr. Reinhold Heckmann.

2.3 A useful theorem

The following lemma and theorem are important since they allow us to restrict the proofs of our extension theorems to finite valuations without loosing generality.

Lemma 2.11 *Let $(X, \mathcal{G})$ be a topological space and ν be a continuous valuation on X. Let $\langle G_j \rangle_{j\in(I,\sqsubseteq)}$ be a directed family of open sets such that $X = \cup^{\uparrow}_{j\in I} G_j$ and $\nu(G_j) < \infty$ for all $j \in I$. Suppose that for each $j \in I$ the restriction $\nu_j(G) := \nu(G \cap G_j)$, extends uniquely to a (finite) Borel measure $\overline{\nu}_j$. Then ν extends uniquely to a Borel measure.*

Proof Let $\overline{\nu} : Cres(X) \to \overline{\mathbb{R}}_+$ be defined as $\overline{\nu}(C) := \sup_{j\in I} \overline{\nu}_j(C)$. We claim that $\overline{\nu}$ is an extension of ν and satisfies the hypothesis of part (2) of Proposition 1.23.

1. Claim: $\overline{\nu}$ is an extension of ν. Suppose that $G \in \mathcal{G}$, then

$$\begin{aligned} \overline{\nu}(G) &= \sup_{j\in I} \overline{\nu}_j(G) \\ &= \sup_{j\in I} \nu(G \cap G_j) \\ &= \nu(G) && \text{since } X = \textstyle\bigcup^{\uparrow}_{j\in I} G_j \text{ and } \nu \text{ is continuous.} \end{aligned}$$

2. Claim: for all $C \in Cres(X)$ the set $\{\overline{\nu}_j(C)\}_{j\in I}$ is $\leq$-directed. Suppose that $C = H_2 \backslash H_1$ with $H_1, H_2 \in \mathcal{G}$ and $H_1 \subseteq H_2$. If $j, k \in I$ with $j \sqsubseteq k$, then

$$\begin{aligned} \overline{\nu}_k(C \cap G_j) &= \overline{\nu}_k(H_2 \cap G_j) - \overline{\nu}_k(H_1 \cap G_j) && \text{since } \overline{\nu}_k \text{ is finite} \\ &= \nu_k(H_2 \cap G_j) - \nu_k(H_1 \cap G_j) \\ &= \nu(H_2 \cap G_j \cap G_k) - \nu(H_1 \cap G_j \cap G_k) \\ &= \nu(H_2 \cap G_j) - \nu(H_1 \cap G_j) && \text{since } G_j \subseteq G_k \\ &= \overline{\nu}_j(H_2) - \overline{\nu}_j(H_1) \\ &= \overline{\nu}_j(C). \end{aligned}$$

   Then note that,

$$\begin{aligned} \overline{\nu}_k(C) &= \overline{\nu}_k(C \cap G_j) + \overline{\nu}_k(C \backslash G_j) \\ &= \overline{\nu}_j(C) + \overline{\nu}_k(C \backslash G_j) \\ &\geq \overline{\nu}_j(C) \end{aligned}$$

3. Claim: $\overline{\nu}$ is finitely additive. Suppose that C_1, C_2, $C_1 \cup C_2 \in Cres(X)$ with $C_1 \cap C_2 = \emptyset$. Since by Lemma 2.2 we know that $+ : \overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ is a Scott continuous function, then

$$\begin{aligned} \overline{\nu}(C_1 \cup C_2) &= \sup_{j\in I} \overline{\nu}_j(C_1 \cup C_2) \\ &= \sup_{j\in I} (\overline{\nu}_j(C_1) + \overline{\nu}_j(C_2)) \\ &= \sup_{j\in I} \overline{\nu}_j(C_1) + \sup_{j\in I} \overline{\nu}_j(C_2) && \text{since both sets are} \\ &= \overline{\nu}(C_1) + \overline{\nu}(C_2) && \text{directed} \end{aligned}$$

4. Claim: $\overline{\nu}$ is σ-subadditive. Since $Cres(X)$ is a semialgebra it suffices to consider the case where the sequence of crescents is disjoint (see Proposition 9 in Section 12.2 of [Roy88]). Suppose that $C \in Cres(X)$ is the union of a disjoint sequence (C_n) in $Cres(X)$. Since $\overline{\nu}(C_n) = \sup_{j\in I} \overline{\nu}_j(C_n)$ it follows that $\overline{\nu}_j(C_n) \leq \overline{\nu}(C_n)$ for all $n \in \mathbb{N}$. Hence $\sum_{n\in\mathbb{N}} \overline{\nu}_j(C_n) \leq \sum_{n\in\mathbb{N}} \overline{\nu}(C_n)$ for all $j \in I$. We conclude that

$$\begin{aligned}\overline{\nu}(C) &= \sup_{j\in I} \overline{\nu}_j(\textstyle\sum_{n\in\mathbb{N}} C_n) \\ &= \sup_{j\in I} \textstyle\sum_{n\in\mathbb{N}} \overline{\nu}_j(C_n) \quad \text{since } \overline{\nu}_j \text{ is a measure} \\ &\leq \textstyle\sum_{n\in\mathbb{N}} \overline{\nu}(C_n)\end{aligned}$$

By part (2) of Proposition 1.23 ν extends uniquely to a measure on $\sigma(Cres(X))$ $= \mathcal{B}X$. ■

Corollary 2.12 *Let $(X, \mathcal{G})$ be a topological space. Every locally finite continuous valuation on X is the supremum of a directed family of finite continuous valuations.*

Proof Let ν be a locally finite continuous valuation on X. By Lemma 2.3 there exists a directed family $\langle G_j\rangle_{j\in I}$ of open sets such that $X = \cup^{\uparrow}_{j\in I} G_j$ and $\nu(G_j) < \infty$ for all $j \in I$. For each $j \in I$ the restriction $\nu_j(G) := \nu(G \cap G_j)$ is finite and continuous. By Claims (1) and (2) in the proof of Lemma 2.11 we have that the family $\{\nu_j\}_{j\in I}$ is directed and $\nu = \bigsqcup^{\uparrow}_{j\in I} \nu_j$. ■

Theorem 2.13 *Let $(X, \mathcal{G})$ be a topological space such that every finite continuous valuation on X extends uniquely to a Borel measure. Then every locally finite continuous valuation on X extends uniquely to a Borel measure.*

Proof Let ν be a locally finite continuous valuation on X. By Lemma 2.3 there exists a directed family $\langle G_j\rangle_{j\in I}$ of open sets such that $X = \cup^{\uparrow}_{j\in I} G_j$ and $\nu(G_j) < \infty$ for all $j \in I$. For each $j \in I$ the restriction $\nu_j(G) := \nu(G \cap G_j)$ is finite and continuous, hence by assumption it extends uniquely to a Borel measure $\overline{\nu}_j$. The conclusion follows immediately from Lemma 2.11. ■

2.4 Extension results for monotone convergence spaces

Now we are ready to present our extension results for monotone convergence spaces. The heart of the extension result is the following choice lemma, for which we give two proofs.

Lemma 2.14 *[AMESD00, Proposition 3.2] Let $(D, \mathcal{G})$ be a monotone convergence space and $\langle \nu_n\rangle_{n\in\mathbb{N}}$ be an ω-chain of simple valuations in $\mathbf{P}^1 D$. Let (A_n) be a sequence of pairwise disjoint sets such that $A_n \subseteq |\nu_n|$ and $\nu_n(A_n) \geq \alpha$ for some fixed $\alpha > 0$ and for all $n \in \mathbb{N}$. Then there exists an ω-chain $\langle x_n\rangle_{n\in\mathbb{N}}$ in D such that $x_n \in A_n$ for infinitely many $n \in \mathbb{N}$.*

Proof See Sections 2.5 and 2.6. ■

We show that if the $\langle \nu_n\rangle_{n\in\mathbb{N}}$ are "independent", then this lemma becomes an easy consequence of the Axiom of Choice or the Borel-Cantelli Lemma. By

$\langle \nu_n \rangle_{n \in \mathbb{N}}$ being independent we mean that $b \sqsubseteq c$ for all $(b,c) \in |\nu_n| \times |\nu_{n+1}|$ for all $n \in \mathbb{N}$. In particular, $b \sqsubseteq c$ for all $(b,c) \in A_n \times A_{n+1}$. Since $\overline{\nu}_n(A_n) > 0$, then $A_n \neq \emptyset$. The Axiom of Choice implies that the Cartesian product $\Pi_n A_n \neq \emptyset$. Any $(x_n) \in \Pi_n A_n$ corresponds to an ω-chain in D with the desired property. Alternatively, let $(\Omega, \Gamma, \mathbb{P})$ be the Cartesian product (see Section 1.8) of the probability spaces $(D, \mathcal{B}D, \overline{\nu}_n)$. Then, by the remarks following Lemma 2.4 above, $\mathbb{P}(\langle A_n \rangle) = \overline{\nu}_n(A_n)$ and $\mathbb{P}(\langle A_{n_1} \rangle \cap \cdots \cap \langle A_{n_k} \rangle) = \Pi_{j=1}^k \mathbb{P}(\langle A_{n_j} \rangle)$ for all $\{n_1, \ldots n_k\} \subseteq_f \mathbb{N}$. Hence $(\langle A_n \rangle)$ is a sequence of independent events such that $\mathbb{P}(\langle A_n \rangle) \geq \alpha > 0$; therefore $\sum_{n \in \mathbb{N}} \mathbb{P}(\langle A_n \rangle) = \infty$. By the Borel-Cantelli Lemma (Proposition 1.22) we have that $\mathbb{P}(\limsup_n A_n) = 1$. Any element (x_n) in $\limsup_n A_n$ corresponds to an ω-chain in D such that $x_n \in A_n$ infinitely often. The proof in Section 2.5 extends this argument for the case that the $\langle \nu_n \rangle_{n \in \mathbb{N}}$ are "non-independent". Although this is proved directly, there exist extended versions of the Borel-Cantelli Lemma for non-independent events (see [Loè51, Lemma 3.5] or [Nas54]) which also imply this result. There is also some resemblance with the following result by M. E. Rudin,

Proposition 2.15 *(Rudin's Lemma) Let $(P, \sqsubseteq)$ be a poset. Suppose that $\{F_j\}_{j \in J}$ is a family of nonempty finite subsets of P such that the paving $\{\uparrow F_j\}_{j \in J}$ is filtered (i.e. $\supseteq$-directed). Then there exists a directed subset $D \subseteq \bigcup F_j$ such that $D \cap F_j \neq \emptyset$ for all $j \in J$.*

Proof See Theorem 4.11 in [Jun89]. The original version of the result can be found in [Rud80]. ■

Unfortunately, in Lemma 2.14 we cannot guarantee that the paving $\{\uparrow A_n\}_{n \in \mathbb{N}}$ is filtered, instead we have a weighting hypothesis which asserts that $\nu_n(A_n) \geq \alpha$ for some $\alpha > 0$. Rudin's Lemma belongs to the long list of implications of the Axiom of Choice. The proof in Section 2.5 uses Kolmogorov's Theorem, whose proof in turn relies on Tychonoff's Theorem. The proof in Section 2.6 uses König's Lemma (Proposition 1.1). The moral seems to be that no matter what we do, at the end of the day, everything relies on the Axiom of Choice.

We start by showing our extension result for normalised valuations. The next theorem holds the thread of the proof, the main results are corollaries of it.

Theorem 2.16 *[AMESD00, Theorem 4.1] Let $(D, \mathcal{G})$ be a monotone convergence space and $\langle \mu_j \rangle_{j \in J}$ be a directed set of simple valuations in $\mathbf{P}^1 D$ with $\mu = \sup_{j \in J} \mu_j$. Then μ has a unique extension to a Borel measure.*

Proof Since μ is the supremum of a directed family of continuous valuations, μ is a continuous valuation and $\mu(D) = 1$. In order to apply Corollary 1.25 we only need to check that the extension given by $\overline{\mu}(U \backslash V) := \mu(U) - \mu(V)$ for $U, V \in \mathcal{G}$ and $V \subseteq U$ is σ-subadditive on $Cres(D)$.

Suppose that $T \in Cres(D)$ is the union of a countable disjoint sequence (T_i) in $Cres(D)$. We show that $\overline{\mu}(T) \leq \sum_{i \in \mathbb{N}} \overline{\mu}(T_i)$. In order to derive a contradiction, assume that $\lim_m \sum_{i=1}^{m} \overline{\mu}(T_i) = \sum_{i \in \mathbb{N}} \overline{\mu}(T_i) < \overline{\mu}(T)$. This happens if and only if there exists some $\alpha > 0$ such that for all $m \in \mathbb{N}$ we have that $\sum_{i=1}^{m} \overline{\mu}(T_i) \leq \overline{\mu}(T) - 2\alpha$. Therefore $2\alpha \leq \lim_j \mu_j(\bigcup_{i>m} T_i)$. This means that if we fix $m \in \mathbb{N}$ there exists $n_m \in J$ such that $n_m \sqsubseteq j$ implies that $\mu_j(\bigcup_{i>m} T_i) \geq \alpha$.

Let $n_0 \in J$ be such that $\mu_{n_0}(T) \geq \alpha$. Observe that for all $j \in J$, the set $|\mu_j| \cap T$ is finite. In particular, since $|\mu_{n_0}|$ is finite, there exists $p_1 \in \mathbb{N}$ such that $|\mu_{n_0}| \cap T \subseteq \bigcup_{m \leq p_1} T_m$. Since we are assuming that $\lim_{j \in J} \mu_j(\bigcup_{m>p_1} T_m) \geq 2\alpha$, there exists $n_1 \in J$ such that $\mu_{n_1}(\bigcup_{m>p_1} T_m) \geq \alpha$. Because $\langle \mu_j \rangle_{j \in J}$ is directed, we can also assume that $\mu_{n_0} \sqsubseteq \mu_{n_1}$. In general, since $|\mu_{n_j}| \cap T \subseteq \bigcup_{m \leq p_{j+1}} T_m$ is finite we can repeat the argument and find $n_{j+1} \in J$ and $p_{j+2} \in \mathbb{N}$ such that

- $\mu_{n_j} \sqsubseteq \mu_{n_{j+1}}$;
- $|\mu_{n_{j+1}}| \cap T \subseteq \bigcup_{m \leq p_{j+2}} T_m$;
- $A_{j+1} = |\mu_{n_{j+1}}| \cap \bigcup_{p_{j+1} < m \leq p_{j+2}} T_m$ with $\mu_{n_{j+1}}(A_{j+1}) \geq \alpha$.

Note that the sets A_j are pairwise disjoint for all $j \in \mathbb{N}$. Then $\langle \mu_{n_k} \rangle_{k \in \mathbb{N}}$ is an ω-chain of normalised simple valuations with $\mu_{n_k}(A_k) \geq \alpha$. Applying Lemma 2.14 we know that there exists an ω-chain $\langle x_k \rangle_{k \in \mathbb{N}}$ in D such that $x_k \in A_k$ for infinitely many $k \in \mathbb{N}$. Since we can remove from this chain all the terms x_k such that $x_k \notin A_k$ we can assume, without loss of generality, that each x_k belongs to one and only one A_n. Since T is a crescent, by part (2) of Lemma 2.8 it follows that $\sqcup^{\uparrow}_{k \in \mathbb{N}} x_k \in T$. Hence there exists i_0 such that $\sqcup^{\uparrow}_{k \in \mathbb{N}} x_k \in T_{i_0}$. By part (3) of Lemma 2.8, an infinite number of x_k will be in T_{i_0} and this is a contradiction since each x_k was chosen in a different T_i and the T_i are pairwise disjoint.

We conclude that $\overline{\mu}$ is σ-subadditive in $Cres(D)$. By Corollary 1.25 we conclude that μ has a unique extension to a measure on the σ-algebra generated by $\mathcal{G}$. ■

Problem 2.17 *In the previous theorem is the measure $\overline{\mu}$ compact or per-*

fect? (See Sections 3.2 and 4.2 for definitions and Corollaries 3.40 and 4.15 for positive results in this direction)

Although the reader might feel tempted to give a more direct proof for the last theorem by repeating the argument used in the proof of Lemma 2.11, we note that in general, for a crescent $C \in Cres(D)$, the net $(\overline{\mu}_j(C))_{j \in J}$ is not a directed subset of $(\overline{\mathbb{R}}_+, \leq)$.

Remark 2.18 ***On countability and projective systems.*** *As we observed before, the cornerstone in the proof of Theorem 2.16 is Lemma 2.14. Although the theorem gives a general result for any directed family of simple valuations the latter lemma relies on a countable argument. This is not surprising if we think that countability is deeply ingrained in the definitions of σ-algebra and measure; as opposed to valuations which are purely a topological concept. In Section 4.2 we discuss projective systems of measure spaces. In the work of Musiał [Mus80, Section 4] sequential convergence (that is, convergence of sequences of measure spaces) plays a key role in the existence of projective limits of perfect measures. A similar situation occurs in the work of Frolík [Fro72, Theorem 2.1] with simple subpresheaves and Bochner's condition. See also Remark 4.5 below. Notice that if $(D, \mathcal{G})$, μ and $\langle \mu_j \rangle_{j \in J}$ are as in the previous theorem, then $(D, \mathcal{B}D, \overline{\mu}_j, Id)_J$ "looks like" a projective system with target space $(D, \mathcal{B}D)$. The difference lies in that we only know $\overline{\mu}_j(G) \geq \overline{\mu}_i(G)$ for all $G \in \mathcal{G}$ and $i \sqsubseteq j$ (compare with Definition 4.3).*

Problem 2.19 *Suppose that $(D, \mathcal{G})$ is a monotone convergence space and $\langle \mu_j \rangle_{j \in J}$ is a directed set in $\mathbf{P}^1 D$ with $\mu = \sup_{j \in J} \mu_j$.*

1. *If μ_j extends uniquely to a Borel measure $\overline{\mu}_j$ for all $j \in J$, then is this the case for μ? What if $\overline{\mu}_j$ is inner regular for all $j \in J$?*

2. *Can we give a suitable definition of p_{ji} for $i, j \in J$ with $i \sqsubseteq j$ such that $(D, \mathcal{B}D, \overline{\mu}_j, p_{ji})_J$ becomes a (weaker notion of) projective system of measures and $(D, \mathcal{B}D, \overline{\mu})$ is the limit space?*

3. *Can we do the same for the more restricted setting in Lemma 2.11?*

In light of Lemma 2.6, if $\mu_i \sqsubseteq \mu_j$ are simple valuations we could set a projection in the power set of D as $p_{ji}(E) := \{x \in |\mu_i| \mid x \sqsubseteq y \text{ and } y \in E \cap |\mu_j|\}$. For a general definition we will probably need the generalised splittings defined in Section 7.5, or some result related to disintegration or factorization (see Section 4.4); or restrict to cases where an embedding projection pair (see Section 3.1.4 in [AJ94] for example) (e_{ij}, p_{ji}) exists from $|\mu_i|$ to $|\mu_j|$.

Corollary 2.20 *Let $(D, \mathcal{G})$ be a monotone convergence space and $\langle \mu_j \rangle_{j \in J}$ be a directed set of simple valuations on D with $\mu = \sup_{j \in J} \mu_j$. If $\mu(D) < \infty$, then μ has a unique extension to a Borel measure.*

Proof We show that this case reduces to the normalised case. First observe that if $\mu(D) = 0$, then μ is identically zero; therefore the constant zero measure is the only possible extension of μ. If $\mu(D) > 0$, let $\mu' := \mu/\mu(D)$ and $\mu'_j := \mu_j/\mu(D)$. Then $\mu'(D) = 1$ and $\mu' = \sup_{j \in J} \mu'_j$ where $\langle \mu'_j \rangle_{j \in J}$ is a directed set of simple valuations. It is clear that μ will have a unique extension to a measure if and only if μ' does. We consider the lifting $D_\perp := D \cup \{\perp\}$ where $\perp$ is the least element of $D_\perp$ and $\perp \notin D$. Then we have that $\mathcal{G}_\perp = \mathcal{G} \cup \{D_\perp\}$. Since $\mu'(D) = 1$, it is clear that we can extend μ' to $D_\perp$ by letting $\mu'(D_\perp) = 1$. For all $j \in J$, let

$$\nu_j := \mu'_j + (1 - \mu'_j(D))\, \delta_\perp.$$

Then ν_j is a simple valuation, $\nu_j(D_\perp) = \nu_j(\uparrow\perp) = 1$ and $\nu_j(G) = \mu'_j(G)$ for all $G \in \mathcal{G} \backslash \{D_\perp\}$. Therefore $\langle \nu_j \rangle_{j \in J}$ is a set of normalised simple valuations on $D_\perp$ with $\mu' = \sup_{j \in J} \nu_j$ and the conclusion follows from Theorem 2.16. Now we can apply Theorem 2.16 and find a unique extension of μ' to a measure $\overline{\mu}'$ on $\mathcal{B}D_\perp$. The restriction of this measure to $\mathcal{B}D$ will give us the desired measure since $\overline{\mu}'(\{\perp\}) = \overline{\mu}'(D_\perp \backslash D) = \mu'(D_\perp) - \mu'(D) = 0$. ■

With the same hypothesis as in Corollary 2.20 suppose, more generally, that $\mu(D) = \infty$. If D has a least element $\perp$, then the set $\{\perp\}$ is closed. Therefore $D' = D \backslash \{\perp\} \in \mathcal{G}$. Hence $\mu(D') < \infty$ or $\mu(D') = \infty$. In the first case it is straightforward to check that $\mu \restriction_{D'} = \sup_{j \in J} \mu_j \restriction_{D'}$; then by Corollary 2.20 $\mu \restriction_{D'}$ can be extended uniquely to a Borel measure on D'. In order to extend this measure to the whole space D we only need to assign some weight to $\{\perp\}$. Since a measure is finitely additive and we are assuming that $\mu(D') < \infty$ the only possible choice is to set $\overline{\mu}(\{\perp\}) := \infty$. So if $\mu(D') < \infty$ the extension is unique. If $\mu(D') = \infty$ and $\mu \restriction_{D'}$ has a unique extension to a Borel measure on D', $\overline{\mu}(\{\perp\})$ can be assigned any nonnegative extended real value and this function will still be a measure; therefore the extension to D is not unique in this case.

The following is the main theorem of the chapter (generalises Theorem 5.1 in [AMESD00]).

Theorem 2.21 ***Extension Theorem for Monotone Convergence Spaces.*** *Let $(D, \mathcal{G})$ be a monotone convergence space and $\langle \mu_j \rangle_{j \in J}$ be a directed* *set of simple valuations on D with $\mu = \sup_{j \in J} \mu_j$. If μ is locally finite, then μ extends uniquely to a τ-smooth Borel measure.*

Proof First notice that μ is continuous since it is the directed supremum of a set of continuous valuations. By Lemma 2.3 there exists a directed family $\langle G_i \rangle_{i \in I}$ of open sets such that $D = \cup^{\uparrow}_{i \in I} G_i$ and $\mu(G_i) < \infty$ for all $i \in I$. For each $i \in I$ the restriction $\mu^i(G) := \mu(G \cap G_i)$ is finite and continuous. Since $\mu^i = \sup_{j \in J} \mu^i_j$ where $\mu^i_j(G) := \mu_j(G \cap G_i)$ is a simple valuation for all $j \in J$, then by Corollary 2.20 it extends uniquely to a finite Borel measure $\overline{\mu}^i$. The conclusion follows from Lemma 2.11. ■

Corollary 2.22 ***Extension Theorem for Continuous dcpo's***

If D is a continuous dcpo, then every locally finite continuous valuation extends uniquely to a τ-smooth measure on the Borel σ-algebra of $(D, \mathcal{G}_\sigma D)$.

Proof If D is a continuous dcpo, then by Proposition 2.5 every continuous valuation on D is the supremum of a directed set of simple valuations. The conclusion follows immediately from Theorem 2.21. ■

Remark 2.23 *These results generalise Corollary 4.3 and Theorem 5.1 in [AMESD00] which were only given for finite and σ-finite valuations. Theorem 3.27 in the next chapter gives another proof for the last corollary.*

We can also give a short proof for a result by Norberg.

Corollary 2.24 *[Nor89, Theorem 3.9] Let D be a continuous dcpo with a second countable Scott topology. A function $\mu : \mathcal{G}_\sigma \to [0, \infty]$ that is finite on the set $\{U \in \mathcal{G}_\sigma \mid U \ll D\}$ has a unique extension to a Borel measure if and only if it is a countably continuous valuation.*

Proof The 'only if' part is immediate since the restriction of a Borel measure to the open sets gives a countably continuous valuation. For the 'if' part notice that since D is a continuous dcpo with second countable Scott topology then, by Lemma 2.9, D is an ω-continuous dcpo. In an ω-continuous dcpo, countably continuous valuations are continuous by Lemma 2.10. Hence μ is a continuous valuation. Let $B := \{b_n \mid n \in \mathbb{N}\}$ be a countable basis for D. We have that $\Uparrow b_n \ll D$ for all $n \in \mathbb{N}$ since if $D = \cup^{\uparrow} \mathcal{G}^*$ with $\mathcal{G}^* \subseteq \mathcal{G}$, then there must exist $G_n \in \mathcal{G}^*$ such that $b_n \in G_n$; hence $\Uparrow b_n \subseteq G_n$. By our assumptions it follows that $\mu(\Uparrow b_n) < \infty$ for all $n \in \mathbb{N}$ and $D = \bigcup_{n \in \mathbb{N}} \Uparrow b_n$. Since the family $\{\bigcup_{k=1}^m \Uparrow b_k \mid m \in \mathbb{N}\}$ is directed, then by Lemma 2.3 μ is locally finite. The conclusion follows from Theorem 2.21. ■

2.5 First proof for Lemma 2.14

Let $(D, \mathcal{G})$ be a monotone convergence space and $\langle \nu_n \rangle_{n \in \mathbb{N}}$ be an ω-chain of normalised simple valuations on D. Consider a sequence of functions $f_i : |\nu_i| \times |\nu_{i+1}| \to [0,1]$ such that f_i satisfies the conditions of Lemma 2.4 for ν_i and ν_{i+1}. Let $S := \Pi_i |\nu_i|$ endowed with the product topology of the discrete spaces $|\nu_i|$. Recall that for any $A \subseteq B \subseteq \mathbb{N}$, we have that $\pi_{BA} : \Pi_{i \in B} |\nu_i| \to \Pi_{i \in A} |\nu_i|$ is the natural projection of $\Pi_{i \in B} |\nu_i|$ onto $\Pi_{i \in A} |\nu_i|$. For any $E \subseteq S$ and $n \in \mathbb{N}$ define $E \restriction_n := \pi_{\mathbb{N}\{0,1,\ldots,n\}}(E)$, i.e. the projection of the elements of E into their first $n+1$ coordinates. Let $S_{[j,j+n]} := \Pi_{j \leq i \leq j+n} |\nu_i|$. For a given $s := (x_j, x_{j+1}, \ldots, x_{j+n}) \in S_{[j,j+n]}$ we define

$$P_{[j,j+n]}(\{s\}) := \overline{\nu}_j(\{x_j\}) \prod_{j \leq i < j+n} \frac{f_i(x_i, x_{i+1})}{\overline{\nu}_i(\{x_i\})}.$$

Each factor $f_i(x, y)/\overline{\nu}_i(\{x\})$ gives the conditional probability that $y \in |\nu_{i+1}|$, given that $x \in |\nu_i|$. According to this interpretation, $P_{[j,j+n]}(\{s\})$ is the probability of the orbit s. Since the set $S_{[j,j+n]}$ is finite, then for all $E \subseteq S_{[j,j+n]}$ we can define $P_{[j,j+n]}(E) := \sum_{s \in E} P_{[j,j+n]}(\{s\})$ and the following lemma holds.

Lemma 2.25 *(Composition of splittings) Let j, k, m, n be elements of $\mathbb{N}$.*

1. *If $j \leq m \leq j+n$, then $\overline{\nu}_m = P_{[j,j+n]} \circ \pi^{-1}_{\{j,\ldots,j+n\}\{m\}}$*
2. *$P_{[n,n+1]}(\{(x, y)\}) = f_n(x, y)$.*
3. *If $j \leq k \leq m \leq n$, then $P_{[k,m]} = P_{[j,n]} \circ \pi^{-1}_{\{j,\ldots,n\}\{k,\ldots,m\}}$*

Proof

1. Let B be any subset of $|\nu_m|$. The Lemma's assertion corresponds to the intuitive idea that the probability of all orbits intersecting B is the same as the probability of B. First we prove the result for the case $m = j + n$ by induction over n.

   - For $n = 0$ the result is trivial since $B \subseteq |\nu_j|$ and $P_{[j,j]}(\pi^{-1}_{\{j\}\{j\}} B) = P_{[j,j]}(B) = \sum_{s \in B} P_{[j,j]}(\{s\}) = \sum_{s \in B} \overline{\nu}_j(\{s\}) = \overline{\nu}_j(B)$.
   - Now suppose that the result holds for $k-1$ and let $n := k$. For all $b \in B$ we have that,

     $P_{[j,j+k]}(S_{[j,j+k-1]} \times \{b\})$

$$\begin{aligned}
&= P_{[j,j+k]}(S_{[j,j+k-2]} \times |\nu_{j+k-1}| \times \{b\}) \\
&= \textstyle\sum_{a\in|\nu_{j+k-1}|} P_{[j,j+k]}(S_{[j,j+k-2]} \times \{a\} \times \{b\}) \\
&= \textstyle\sum_{a\in|\nu_{j+k-1}|} P_{[j,j+k-1]}(S_{[j,j+k-2]} \times \{a\}) \frac{f_{j+k-1}(a,b)}{\overline{\nu}_{j+k-1}(\{a\})} \\
&= \textstyle\sum_{a\in|\nu_{j+k-1}|} f_{j+k-1}(a,b) \\
&\qquad\qquad \text{by inductive hypothesis} \\
&= \overline{\nu}_{j+k}(\{b\}) \\
&\qquad\qquad \text{by Lemma 2.4.}
\end{aligned}$$

Since $\pi^{-1}_{\{j,\ldots,j+k\}\{j+k\}}B$ is the disjoint union of the sets $S_{[j,j+k-1]} \times \{b\}$ with $b \in B$ we conclude that $P_{[j,j+k]}(\pi^{-1}_{\{j,\ldots,j+k\}\{j+k\}}B) = \sum_{b\in B} \overline{\nu}_{j+k}(\{b\}) = \overline{\nu}_{j+k}(B)$.

In the same way it is possible to prove the case $m = j$; that is $P_{[j,j+n]}(\pi^{-1}_{\{j,\ldots,j+n\}\{j\}}B) = \overline{\nu}_j(B)$ for all $n \in \mathbb{N}$. Now suppose that $j < m < j+n$. Notice that for all $(x_j,\ldots,x_{j+n}) \in S_{[j,j+n]}$ we have that

$$P_{[j,j+n]}(\{(x_j,\ldots,x_m,\ldots,x_{j+n})\}) = \frac{P_{[j,m]}(\{(x_j,\ldots,x_m)\})P_{[m,j+n]}(\{(x_m,\ldots,x_n)\})}{\overline{\nu}_m(\{x_m\})}$$

Fix $b \in B$ and let $C := S_{[j,m-1]}$ and $D := S_{[m+1,j+n]}$ then

$$\begin{aligned}
P_{[j,j+n]}(C \times \{b\} \times D) &= \textstyle\sum_{c\in C}\sum_{d\in D} P_{[j,j+n]}(\{(c,b,d)\}) \\
&= \textstyle\sum_{c\in C}\sum_{d\in D} \frac{P_{[j,m]}(\{(c,b)\})P_{[m,j+n]}(\{(b,d)\})}{\overline{\nu}_m(\{b\})} \\
&= \textstyle\sum_{c\in C} \frac{P_{[j,m]}(\{(c,b)\})}{\overline{\nu}_m(\{b\})} \sum_{d\in D} P_{[m,j+n]}(\{(b,d)\}) \\
&= \textstyle\sum_{c\in C} P_{[j,m]}(\{(c,b)\}) \\
&\qquad\qquad \text{by the previous results} \\
&= \overline{\nu}_m(\{b\}) \\
&\qquad\qquad \text{by the previous results}
\end{aligned}$$

Once again since $\pi^{-1}_{\{j,\ldots,j+n\}\{m\}}B$ is the disjoint union of the sets $C \times \{b\} \times D$ with $b \in B$ we conclude that $P_{[j,j+n]}(\pi^{-1}_{\{j,\ldots,j+n\}\{m\}}B) = \sum_{b\in B} \overline{\nu}_m(\{b\}) = \overline{\nu}_m(B)$.

2. This is immediate from the definition of $P_{[n,n+1]}$.

3. Let E be any subset of $S_{[j+1,n]}$; then

   $P_{[j,n]} \circ \pi^{-1}_{\{j,j+1,\ldots,n\}\{j+1,\ldots,n\}}(E)$

$$
\begin{aligned}
&= P_{[j,n]}(|\nu_j| \times E) \\
&= \textstyle\sum_{(x_{j+1},\ldots,x_n)\in E} \sum_{x_j\in|\nu_j|} P_{[j,n]}(\{(x_j, x_{j+1}, \ldots, x_n)\}) \\
&= \textstyle\sum_{(x_{j+1},\ldots,x_n)\in E} \sum_{x_j\in|\nu_j|} f_j(x_j, x_{j+1}) \prod_{j+1\leq i<n} \frac{f_i(x_i,x_{i+1})}{\overline{\nu}_i(\{x_i\})} \\
&= \textstyle\sum_{(x_{j+1},\ldots,x_n)\in E} \left(\prod_{j+1\leq i<n} \frac{f_i(x_i,x_{i+1})}{\overline{\nu}_i(\{x_i\})}\right) \sum_{x_j\in|\nu_j|} f_j(x_j, x_{j+1}) \\
&= \textstyle\sum_{(x_{j+1},\ldots,x_n)\in E} \left(\prod_{j+1\leq i<n} \frac{f_i(x_i,x_{i+1})}{\overline{\nu}_i(\{x_i\})}\right) \overline{\nu}_{j+1}(\{x_{j+1}\}) \\
&\qquad\qquad \text{by Lemma 2.4} \\
&= \textstyle\sum_{(x_{j+1},\ldots,x_n)\in E} P_{[j+1,n]}(\{(x_{j+1}, \ldots, x_n)\}) \\
&= P_{[j+1,n]}(E)
\end{aligned}
$$

By induction we can show that for any $E \subseteq S_{[k,n]}$

$$
\begin{aligned}
P_{[j,n]} \circ \pi^{-1}_{\{j,\ldots,n\}\{k,\ldots,n\}}(E) &= P_{[j,n]}(S_{[j,k-1]} \times E) \\
&= P_{[k,m]}(E)
\end{aligned}
$$

The rest of the claim follows by a similar argument. ■

In particular $P_{[j,j+n]}(S_{[j,j+n]}) = \nu_j(|\nu_j|) = 1$; therefore Lemma 2.25 asserts that $\{P_{[j,j+n]}\}_{j,n\in\mathbb{N}}$ is a consistent family of probability measures. Now we show that the functions $P_{[j,j+n]}$ extend to a probability measure on the whole product space S, as was done in Section 1.8 for the product measure (the technique is quite standard [Bil65, Example 1.2][Mañ87, p. 63]). First, we need to define consistent probability measures on $\Pi_{i\in A}|\nu_i|$ for the remaining $A \subseteq_f \mathbb{N}$. Let A denote any finite subset of $\mathbb{N}$, we define a probability measure on $\Pi_{i\in A}|\nu_i|$ by $P_A := P_{[0,max(A)]} \circ \pi^{-1}_{\{0,1,\ldots,max(A)\}A}$. If B is any other finite subset of $\mathbb{N}$ such that $B \subseteq A$, then

$$
\begin{aligned}
P_B &= P_{[0,max(B)]} \circ \pi^{-1}_{\{0,\ldots,max(B)\}B} && \text{by def.} \\
&= P_{[0,max(A)]} \circ \pi^{-1}_{\{0,\ldots,max(A)\}\{0,\ldots,max(B)\}} \circ \pi^{-1}_{\{0,\ldots,max(B)\}B} && \text{by 2.25-(3)} \\
&= P_{[0,max(A)]} \circ \pi^{-1}_{\{0,\ldots,max(A)\}B} \\
&= P_{[0,max(A)]} \circ \pi^{-1}_{\{0,\ldots,max(A)\}A} \circ \pi^{-1}_{AB} && \text{since } B \subseteq A \\
&= P_A \circ \pi^{-1}_{AB} && \text{by def.}
\end{aligned}
$$

Therefore, $\{P_F \mid F \subseteq_f \mathbb{N}\}$ is a consistent family of probability distributions; obviously $P_{\{j,\ldots,j+n\}} = P_{[j,j+n]}$. For all $n \in \mathbb{N}$ the set $|\nu_n|$ is finite hence all its subsets are compact (with respect to any topology) therefore $\overline{\nu}_n$ is inner regular on $|\nu_n|$. So we can apply Kolmogorov's Theorem (Proposition 1.20) and conclude that there exists a probability measure $\mathbb{P}$ on the Borel σ-algebra $\mathcal{B}S$ satisfying $P_F = \mathbb{P} \circ \pi^{-1}_{\mathbb{N}F}$ for all finite $F \subseteq \mathbb{N}$. Recall that if $A_i \subseteq |\nu_i|$ for $i = j,\ldots, j+n$, then

$$\begin{array}{rcl}\langle A_j, \ldots, A_{j+n}\rangle & := & \pi^{-1}_{\mathbb{N}\{j,\ldots,j+n\}}(A_j \times \cdots \times A_{j+n}) \\ & = & \{(x_m) \in S \mid x_{j+i} \in A_{j+i} \text{ for } i = 0, \ldots, n\};\end{array}$$

and the following equalities hold,

- $\mathbb{P}(\langle A_n\rangle) = \nu_n(A_n)$
- $\mathbb{P}(\langle \{x\}_n, \{y\}_{n+1}\rangle) = f_n(x, y)$
- $\mathbb{P}(\langle A_j,..., A_{j+n}\rangle) = P_{\{j,...,j+n\}}(A_j \times \cdots \times A_{j+n})$

A more lengthy proof for the existence of $\mathbb{P}$ that does not use Kolmogorov's Theorem could be given as follows. Recall that the set of cylinders L is a semialgebra and a base for the product topology on S; hence any finite Borel measure on S is uniquely determined by its value on L. Define the function $\mathbb{P}$ on L as $\mathbb{P}(\langle A_j,..., A_{j+n}\rangle) := P_{\{j,...,j+n\}}(A_j \times \cdots \times A_{j+n})$. Show that $\mathbb{P}$ is additive and σ-subadditive, then apply part (2) of Proposition 1.23.

For all $E \subseteq S$ and $n \in \mathbb{N}$ define

$$\begin{array}{rcl}[n, E] & := & \pi^{-1}_{\mathbb{N}\{0,\ldots,n\}}(\pi_{\mathbb{N}\{0,\ldots,n\}}[E]) \\ & = & \{(x_i) \in S \mid (x_0, x_1, \ldots, x_n) \in E\restriction_n\}.\end{array}$$

Note that $[n, E] \supseteq [n + 1, E] \supseteq E$ for all $n \in \mathbb{N}$.

Lemma 2.26

1. *For every $E \subseteq S$, $cl(E) = \bigcap_{n\in\mathbb{N}}[n, E]$.*
2. *The set $C := \{(x_k) \in S \mid x_k \sqsubseteq x_{k+1}$ for all $k \in \mathbb{N}\}$ of chains in S is closed and $\mathbb{P}(C) = 1$.*

Proof

1. Since the space $\Pi_{0\leq i\leq n}|\nu_i|$ is discrete, then $\pi_{\mathbb{N}\{0,\ldots,n\}}(E)$ is a closed set for all $n \in \mathbb{N}$. Hence $[n, E]$ is closed, since it is the inverse image of a closed set under a continuous function. It follows that $\bigcap_{n\in\mathbb{N}}[n, E]$ is closed. Consider any (x_k) in $\bigcap_{n\in\mathbb{N}}[n, E]$. If (x_k) belongs to a cylinder $\langle A_j,..., A_{j+n}\rangle$, then $x_i \in A_i$ for $i = j,..., j + n$. But since $(x_k) \in [j + n, E]$, there exists $(y_k) \in E$ such that $y_i = x_i$ for all $i = 0,..., j + n$; therefore $(y_k) \in \langle A_j,..., A_{j+n}\rangle$. Since the set of cylinders is a basis for the product topology in S, we conclude that $(x_k) \in cl(E)$. Since $\bigcap_{n\in\mathbb{N}}[n, E]$ is closed and contained in $cl(E)$ the claim follows.

2. Notice that $[n, C] = \{(x_k) \in S \mid x_k \sqsubseteq x_{k+1}$ for $k = 0,\dots,n-1\}$. Clearly $C = \bigcap_{n\in\mathbb{N}}[n, C]$ since a sequence is a chain if and only if all its finite initial segments are chains; hence C is closed by part (1). Since $f_k(x, y) > 0$ only if $x \sqsubseteq y$, it follows that $P_{\{0,\dots,n\}}(C \restriction_n) = 1$ for all $n \in \mathbb{N}$. But $\mathbb{P}$ was constructed in such a way that $P_{\{0,\dots,n\}}(C \restriction_n) = \mathbb{P}(\pi^{-1}_{\mathbb{N}\{0,\dots,n\}}(C \restriction_n)) = \mathbb{P}([n, C])$; therefore $([n, C])$ is a decreasing sequence of sets with $\mathbb{P}([n, C]) = 1$. We conclude that $\mathbb{P}(C) = \inf_{n\in\mathbb{N}} \mathbb{P}([n, C]) = 1$. ■

Corollary 2.27 *(Lemma 2.14) Let $(D, \mathcal{G})$ be a monotone convergence space and $\langle\nu_n\rangle_{n\in\mathbb{N}}$ be an ω-chain of simple valuations in $\mathbf{P}^1 D$. Let (A_n) be a sequence of pairwise disjoint sets such that $A_n \subseteq |\nu_n|$ and $\nu_n(A_n) \geq \alpha$ for some fixed $\alpha > 0$ and for all $n \in \mathbb{N}$. Then there exists an ω-chain $\langle x_n\rangle_{n\in\mathbb{N}}$ in D such that $x_n \in A_n$ for infinitely many $n \in \mathbb{N}$.*

Proof Let $C(A_n) := C \cap \langle A_n\rangle$, i.e. the set of chains $(x_k) \in S$ such that $x_n \in A_n$. Since C is closed, it is Borel measurable; therefore

$$\begin{aligned}
\mathbb{P}(C(A_n)) &= \mathbb{P}(C \cap \langle A_n\rangle) \\
&= \mathbb{P}(\langle A_n\rangle) \qquad \text{since } \mathbb{P}(C) = 1 \text{ by the previous lemma} \\
&= \nu_n(A_n) \\
&\geq \alpha.
\end{aligned}$$

Now observe that the subset of chains $(x_k) \in C$ such that $x_n \in A_n$ for infinitely many $n \in \mathbb{N}$, is exactly $\bigcap_{m\in\mathbb{N}} \bigcup_{n\geq m} C(A_n) = \limsup_n C(A_n)$. Therefore $\mathbb{P}(\limsup_n C(A_n)) \geq \alpha > 0$. Since $\mathbb{P}$ is a measure, we conclude that $\limsup_n C(A_n) \neq \emptyset$. ■

2.6 Second proof for Lemma 2.14

Once again let $(D, \mathcal{G})$ be a monotone convergence space and $\langle\nu_n\rangle_{n\in\mathbb{N}}$ be an ω-chain of simple valuations in $\mathbf{P}^1 D$. Let (A_n) be a sequence of pairwise disjoint sets such that $A_n \subseteq |\nu_n|$ and $\nu_n(A_n) \geq \alpha$ for some fixed $\alpha > 0$ and for all $n \in \mathbb{N}$. For the rest of the section the sequence (A_n) and the chain $\langle\nu_n\rangle_{n\in\mathbb{N}}$ remain fixed. We introduce the following definitions.

Definition 2.28 *Fix $j \in \mathbb{N}$. Let $C \subseteq |\nu_j|$ be any subset.*

1. *Define $C^{(j,0)} := C$ and $C^{(j,n+1)} := |\nu_{j+n+1}| \cap {\uparrow} C^{(j,n)}$.*
2. *The members of $C^{(j,n)}$ are called (j, n)-**descendants** of C.*
3. *C is j-**slimming** if $\lim_n \overline{\nu}_{j+n}(C^{(j,n)} \cap A_{j+n}) = 0$.*

4. *C is j-nonslimming if there exists $\varepsilon > 0$ such that for all $n \in \mathbb{N}$ there exists $m \geq n$ such that $\overline{\nu}_{j+m}(C^{(j,m)} \cap A_{j+m}) \geq \varepsilon$. We say that ε is a* ***nonslimming constant*** *for C.*

A set of points is slimming if the sum of the weights of its descendants in A_n tends to zero. Notice that j-nonslimming sets cannot be empty since $\overline{\nu}_n(\emptyset) = 0$ for all $n \in \mathbb{N}$. The aim of the proof is to construct a $\subseteq$-chain of nonslimming sets in order to obtain a finitely branching tree of infinite length whose nodes belong to the sets A_n.

Remark 2.29 *To avoid confusion in what follows, it is good to remember that the notation $C^{(j,n)}$ implies that $C \subseteq |\nu_j|$ and $C^{(j,n)} \subseteq |\nu_{j+n}|$.*

The following claims hold for all $n, m, j, k \in \mathbb{N}$.

1. $C^{(j,n+1)} \subseteq \uparrow C^{(j,n)}$
2. $C^{(j,1)} = |\nu_{j+1}| \cap \uparrow C$, but $C^{(j,2)}$ could be a proper subset of $|\nu_{j+2}| \cap \uparrow C$.
3. **Claim:** $|\nu_j|^{(j,n)} = |\nu_{j+n}|$. Fix $j \in \mathbb{N}$. For $n = 0$ we have that $|\nu_j|^{(j,0)} = |\nu_j|$ by definition. If the claim holds for $n = k$, then $|\nu_j|^{(j,k)} = |\nu_{j+k}|$. By Corollary 2.7, we have that $|\nu_{j+k+1}| \subseteq \uparrow|\nu_{j+k}|$. Therefore $|\nu_j|^{(j,k+1)} := |\nu_{j+k+1}| \cap \uparrow|\nu_j|^{(j,k)} = |\nu_{j+k+1}|$.
4. **Claim:** If $m < n$, then $\overline{\nu}_{j+m}(C^{(j,m)}) \leq \overline{\nu}_{j+n}(C^{(j,n)})$, i.e. the mass of (j,n)-descendants increases with n. It suffices to show that $\overline{\nu}_{j+n}(C^{(j,n)}) \leq \overline{\nu}_{j+n+1}(C^{(j,n+1)})$ for all $n \in \mathbb{N}$. We know that $\nu_{j+n} \sqsubseteq \nu_{j+n+1}$ since $\langle \nu_n \rangle_{n \in \mathbb{N}}$ is a chain. For all $n \in \mathbb{N}$, $C^{(j,n)} \subseteq |\nu_{j+n}|$; then by Lemma 2.6,
$$\begin{aligned} \overline{\nu}_{j+n}(C^{(j,n)}) &\leq \overline{\nu}_{j+n+1}(\uparrow C^{(j,n)}) \\ &= \overline{\nu}_{j+n+1}(|\nu_{j+n+1}| \cap \uparrow C^{(j,n)}) \\ &= \overline{\nu}_{j+n+1}(C^{(j,n+1)}). \end{aligned}$$
5. **Claim:** $C^{(j,m+n)} = (C^{(j,m)})^{(j+m,n)}$. That is, the $m+n$-descendants of C are the same as the n-descendants of the m-descendants of C. For $n = 0$ we have that $C^{(j,m+0)} = (C^{(j,m)})^{(j+m,0)}$. Now suppose that the claim holds for $n = k$; then
$$\begin{aligned} C^{(j,m+k+1)} &= |\nu_{j+m+k+1}| \cap \uparrow C^{(j,m+k)} && \text{by definition} \\ &= |\nu_{j+m+k+1}| \cap \uparrow (C^{(j,m)})^{(j+m,k)} && \text{by inductive hyp.} \\ &= (C^{(j,m)})^{(j+m,k+1)} && \text{by definition} \end{aligned}$$
6. **Claim:** For all $B \subseteq C$ we have that $B^{(j,n)} \subseteq C^{(j,n)}$. The claim is true for $n = 0$. If it holds for $n = k$, then $B^{(j,k+1)} = |\nu_{j+k+1}| \cap \uparrow B^{(j,k)} \subseteq |\nu_{j+k+1}| \cap \uparrow C^{(j,k)} = C^{(j,k+1)}$.

7. **Claim:** $(B \cup C)^{(j,n)} = B^{(j,n)} \cup C^{(j,n)}$. For $n = 0$ the claim is trivial; assume that it holds for $n = k$. Then $(B \cup C)^{(j,k+1)} = |\nu_{j+k+1}| \cap \uparrow(B \cup C)^{(j,k)} = |\nu_{j+k+1}| \cap (\uparrow B^{(j,k)} \cup \uparrow C^{(j,k)}) = B^{(j,k+1)} \cup C^{(j,k+1)}$.

8. **Claim:** $|\nu_j|$ is a j-nonslimming set with nonslimming constant α. By claim (3), we know that $|\nu_j|^{(j,m)} = |\nu_{j+m}|$. For all $m \in \mathbb{N}$, we have that $A_{j+m} \subseteq |\nu_{j+m}|$; hence $\overline{\nu}_{j+m}(|\nu_j|^{(j,m)} \cap A_{j+m}) = \overline{\nu}_{j+m}(A_{j+m}) \geq \alpha$.

9. **Claim:** If $U \subseteq |\nu_j|$ is a j-nonslimming set, then $U^{(j,n)}$ is a $(j+n)$-nonslimming set. Let ε be a nonslimming constant for U. In order to derive a contradiction, suppose that there exists $m \in \mathbb{N}$ such that $U^{(j,m)} \subseteq |\nu_{j+m}|$ is a $(j+m)$-slimming set. Let $\delta := \frac{\varepsilon}{2}$ then there exists $N \in \mathbb{N}$ such that $\overline{\nu}_{j+m+n}((U^{(j,m)})^{(j+m,n)} \cap A_{j+m+n}) < \frac{\varepsilon}{2}$ for all $n \geq N$. By claim (5) we have that

$$\overline{\nu}_{j+m+n}(U^{(j,m+n)} \cap A_{j+m+n}) = \overline{\nu}_{j+m+n}((U^{(j,m)})^{(j+m,n)} \cap A_{j+m+n}) < \tfrac{\varepsilon}{2}$$

for all $n \geq N$, contradicting the fact that U is j-nonslimming with nonslimming constant ε.

Lemma 2.30 *If $U \subseteq |\nu_j|$ is j-nonslimming then there exists $m \geq 1$ such that $U^{(j,m)} \cap A_{j+m}$ is a $(j+m)$-nonslimming set.*

Proof We assume that $U^{(j,n)} \cap A_{j+n}$ is a $(j+n)$-slimming set for all $n \geq 1$ and derive a contradiction. This is done by recursively constructing a sequence of sets (C_n) with certain properties.

- **Base case.** Suppose that $\delta > 0$ is a nonslimming constant for U. Then there exists $n_0 \in \mathbb{N}$ such that for $C_0 := U^{(j,n_0)} \cap A_{j+n_0}$ we have that

  $$\delta \leq \overline{\nu}_{j+n_0}(C_0).$$

  Since we are assuming C_0 is $(j+n_0)$-slimming, then there exists $m_1 > 1$ such that

  $$\overline{\nu}_{j+n_0+m}(C_0^{(j+n_0,m)} \cap A_{j+n_0+m}) < \frac{\delta}{2} \cdot \frac{1}{2} \text{ for all } m \geq m_1. \tag{2.1}$$

  Since U is j-nonslimming, we can find $n_1 > m_1$ such that for $C_1 := U^{(j,n_0+n_1)} \cap A_{j+n_0+n_1}$ we have that

  $$\delta \leq \overline{\nu}_{j+n_0+n_1}(C_1).$$

  Since $n_1 > m_1$, then (2.1) implies that

$$\begin{aligned}&\overline{\nu}_{j+n_0+n_1}(C_0^{(j+n_0,n_1)} \cap A_{j+n_0+n_1}) < \tfrac{\delta}{2}\cdot\tfrac{1}{2}\\&\overline{\nu}_{j+n}(C_0^{(j+n_0,n-n_0)} \cap A_{j+n}) < \tfrac{\delta}{2}\cdot\tfrac{1}{2} \text{ for all } n \geq n_0+n_1\end{aligned}$$

Notice that $C_1 \subseteq A_{j+n_0+n_1}$ implies that

$$\begin{aligned}\overline{\nu}_{j+n_0+n_1}(C_0^{(j+n_0,n_1)} \cap C_1) &\leq \overline{\nu}_{j+n_0+n_1}(C_0^{(j+n_0,n_1)} \cap A_{j+n_0+n_1})\\&< \tfrac{\delta}{2}\end{aligned}$$

But we know that $\delta \leq \overline{\nu}_{j+n_0+n_1}(C_1)$; this means that at least $\frac{\delta}{2}$ of the mass in C_1 belongs to points which are not n_1-descendants of C_0.

- **Recursive step.** Let $S_k := \sum_{i=0}^{k} n_i$ and assume that

$$\left.\begin{array}{ll}C_i = U^{(j,S_i)} \cap A_{j+S_i} & \text{for } i=0,\dots,k\\ \delta \leq \overline{\nu}_{j+S_i}(C_i) & \text{for } i=0,\dots,k\\ \overline{\nu}_{j+n}(C_i^{(j+S_i,n-S_i)} \cap A_{j+n}) < \frac{\delta}{2}\cdot\frac{1}{2^{i+1}} & \text{for } n \geq S_{i+1},\ i=0,\dots,k-1\end{array}\right\} \tag{2.2}$$

Since $C_k \subseteq |\nu_{j+S_k}|$ is $(j+S_k)$-slimming we can find $m_{k+1} > 1$ such that

$$\overline{\nu}_{j+S_k+m}(C_k^{(j+S_k,m)} \cap A_{j+S_k+m}) < \frac{\delta}{2}\cdot\frac{1}{2^{k+1}} \text{ for all } m \geq m_{k+1}. \tag{2.3}$$

As U is j-nonslimming, we can find $n_{k+1} > m_{k+1}$ such that for $C_{k+1} := U^{(j,S_{k+1})} \cap A_{j+S_{k+1}}$ we have that

$$\delta \leq \overline{\nu}_{j+S_{k+1}}(C_{k+1}).$$

Since $n_{k+1} > m_{k+1}$, then (2.3) implies that

$$\begin{aligned}&\overline{\nu}_{j+S_{k+1}}(C_k^{(j+S_k,n_{k+1})} \cap A_{j+S_{k+1}}) < \tfrac{\delta}{2}\cdot\tfrac{1}{2^{k+1}}\\&\overline{\nu}_{j+n}(C_k^{(j+S_k,n-S_k)} \cap A_{j+n}) < \tfrac{\delta}{2}\cdot\tfrac{1}{2^{k+1}} \text{ for all } n \geq S_{k+1}.\end{aligned} \tag{2.4}$$

This completes the definition of the sequence (C_n).

Let $L_{k+1} := \bigcup_{i=0}^{k} C_i^{(j+S_i,S_{k+1}-S_i)}$. Then L_{k+1} is the set of descendants of the sets $\{C_i \mid i=0,...,k\}$ in $|\nu_{j+S_{k+1}}|$. As $S_{k+1} \geq S_{i+1}$ for $i=0,...,k$, we have that

$$\begin{aligned}&\overline{\nu}_{j+S_{k+1}}(L_{k+1} \cap C_{k+1})\\&\quad\leq \overline{\nu}_{j+S_{k+1}}(L_{k+1} \cap A_{j+S_{k+1}}) && \text{since } C_{k+1} \subseteq A_{j+S_{k+1}}\\&\quad= \overline{\nu}_{j+S_{k+1}}\left(\textstyle\bigcup_{i=0}^{k} C_i^{(j+S_i,S_{k+1}-S_i)} \cap A_{j+S_{k+1}}\right)\\&\quad\leq \textstyle\sum_{i=0}^{k} \overline{\nu}_{j+S_{k+1}}\left(C_i^{(j+S_i,S_{k+1}-S_i)} \cap A_{j+S_{k+1}}\right)\\&\quad\leq \tfrac{\delta}{2}\textstyle\sum_{i=0}^{k}\tfrac{1}{2^{i+1}} && \text{by (2.2) and (2.4)}\\&\quad< \tfrac{\delta}{2}.\end{aligned}$$

But $\delta \leq \overline{\nu}_{j+S_{k+1}}(C_{k+1})$; therefore, $\frac{\delta}{2} < \overline{\nu}_{j+S_{k+1}}(C_{k+1} \backslash L_{k+1})$. That is, at least $\frac{\delta}{2}$ of the mass in C_{k+1} does not come from (the descendants of) C_i for $i \leq k$. The intuition is that if a point belongs to C_i, then all its descendants become "labeled" with C_i. A point may become labeled by more than one C_i. The important thing to notice is that at step k, previously unlabeled points in $|\nu_{j+S_k}|$ with total mass greater than $\frac{\delta}{2}$ become labeled with C_k. The contradiction comes if we choose $k \in \mathbb{N}$ such that $k > \frac{2}{\delta}$. At step $k+1$ we will have that $k(\frac{\delta}{2}) > 1$ of the mass labeled by some C_i, but the total mass in $|\nu_{j+S_k}|$ is 1. The sensible reader should be convinced with this intuitive argument and skip to the end of the proof; for the skeptic reader the formal argument follows in all its fastidious rigor. Observe that

$$\begin{aligned} L_{k+2} &= \bigcup_{i=0}^{k+1} C_i^{(j+S_i, S_{k+2}-S_i)} \\ &= \bigcup_{i=0}^{k+1} (C_i^{(j+S_i, S_{k+1}-S_i)})^{(j+S_{k+1}, n_{k+2})} && \text{by claim (5)} \\ &= \left(\bigcup_{i=0}^{k+1} C_i^{(j+S_i, S_{k+1}-S_i)}\right)^{(j+S_{k+1}, n_{k+2})} && \text{by claim (7)} \\ &= (C_{k+1} \cup L_{k+1})^{(j+S_{k+1}, n_{k+2})}. \end{aligned}$$

Therefore

$$\begin{aligned} \overline{\nu}_{j+S_{k+2}}(L_{k+2}) &\geq \overline{\nu}_{j+S_{k+1}}(C_{k+1} \cup L_{k+1}) && \text{by claim (4)} \\ &= \overline{\nu}_{j+S_{k+1}}(C_{k+1} \backslash L_{k+1}) + \overline{\nu}_{j+S_{k+1}}(L_{k+1}) \\ &> \tfrac{\delta}{2} + \overline{\nu}_{j+S_{k+1}}(L_{k+1}) \end{aligned}$$

and since $\overline{\nu}_{j+S_1}(L_1) = \overline{\nu}_{j+n_0+n_1}(C_0^{(j+n_0,n_1)}) \geq \overline{\nu}_{j+n_0}(C_0) \geq \delta$, it follows that $\overline{\nu}_{j+S_{k+1}}(L_{k+1}) > k \cdot \frac{\delta}{2}$ for all $k \in \mathbb{N}$. If we choose $k > \frac{2}{\delta}$, then $\overline{\nu}_{j+S_{k+1}}(L_{k+1}) > 1$: a contradiction. ■

Corollary 2.31 *(Lemma 2.14) Let $(D, \mathcal{G})$ be a monotone convergence space and $\langle \nu_n \rangle_{n \in \mathbb{N}}$ be an ω-chain of simple valuations in $\mathbf{P}^1 D$. Let (A_n) be a sequence of pairwise disjoint sets such that $A_n \subseteq |\nu_n|$ and $\nu_n(A_n) \geq \alpha$ for some fixed $\alpha > 0$ and for all $n \in \mathbb{N}$. Then there exists an ω-chain $\langle x_n \rangle_{n \in \mathbb{N}}$ in D such that $x_n \in A_n$ for infinitely many $n \in \mathbb{N}$.*

Proof We build a sequence (U_k) of nonslimming sets by recursion. By claim (8) we know that $|\nu_0|$ is 0-nonslimming. Apply Lemma 2.30 to find $n_0 \geq 1$ such that $U_0 = |\nu_0|^{(0,n_0)} \cap A_{n_0}$ is n_0-nonslimming. Assume that $U_k \subseteq A_{S_k}$ is S_k-nonslimming, again by Lemma 2.30 there exists $n_{k+1} \geq 1$ such that $U_{k+1} = U_k^{(S_k, n_{k+1})} \cap A_{S_{k+1}}$ is S_{k+1}-nonslimming. Hence, for all $k \in \mathbb{N}$

- $U_k \subseteq A_{S_k} \subseteq |\nu_{S_k}|$. Therefore U_k is finite.
- $U_k \neq \emptyset$. Since nonslimming sets cannot be empty by definition.
- $U_k \subseteq A_{S_k}$ with $k \leq S_k < S_{k+1}$. Therefore the U_k are pairwise disjoint.

- $U_{k+1} \subseteq \uparrow U_k$ by claim (1). Hence any point in U_{k+1} is above at least one point in U_k.

Then $U = \bigcup_{k\in\mathbb{N}} U_k$ is a finitely branching tree with infinite cardinality and we can apply König's Lemma (Proposition 1.1) or Rudin's Lemma (Proposition 2.15) to find a chain $\langle x_k \rangle_{k\in\mathbb{N}}$ such that $x_k \in U_k \subseteq A_{S_k}$ for all $k \in \mathbb{N}$. ■

2.7 Impossibility of a general extension result for dcpo's

The question remains as to whether all finite continuous valuations on a dcpo can be extended to a Borel measure. We show that this is not the case. First we need some result regarding sober spaces. Recall from Section 1.4 that a topological space $(X, \mathcal{G})$ is sober if every closed irreducible set is the closure of a unique point. Let $(L, \leq)$ be a complete lattice. A set $\mathcal{H} \subseteq L$ is called a **completely prime filter** if $\bigvee M \in \mathcal{H}$ implies that $\mathcal{H} \cap M \neq \emptyset$ for all $M \subseteq L$. If $(X, \mathcal{G})$ is any topological space, then $(\mathcal{G}, \subseteq)$ is a complete lattice, and for each $x \in X$ its filter of open neighborhoods $\mathcal{N}(x)$ is completely prime. We denote by $pt(\mathcal{G})$ the paving of all completely prime filters (or points) of $\mathcal{G}$. The set $pt(\mathcal{G})$ can be given a topology as follows, let $\mathcal{O}(G) = \{\mathcal{H} \in pt(\mathcal{G}) \mid G \in \mathcal{H}\}$, then the family $\{\mathcal{O}(G)\}_{G\in\mathcal{G}}$ is a topology on $pt(\mathcal{G})$. For any topological space $(X, \mathcal{G})$ define the map $\eta_X : X \to pt(\mathcal{G})$ as $\eta_X(x) := \mathcal{N}(x)$; that is, each point is mapped to its (completely prime) filter of open nbds.

Lemma 2.32 *Let $(X, \mathcal{G})$ be a topological space, then $pt(\mathcal{G})$ and the set of continuous $\{0,1\}$-valuations on X are in one to one correspondence.*

Proof Let $\mathcal{H} \in pt(\mathcal{G})$, we claim that the indicator function $\mathbb{I}_{\mathcal{H}} : \mathcal{G} \to \{0,1\}$ is a continuous $\{0,1\}$-valuation. Strictness and monotonicity are obvious. For modularity observe that if $\mathbb{I}_{\mathcal{H}}(G_1 \cup G_2) = 1$, then $G_1 \cup G_2 \in \mathcal{H}$ and since $\mathcal{H}$ is prime we have that $G_1 \in \mathcal{H}$ or $G_2 \in \mathcal{H}$. If both belong to $\mathcal{H}$, then $G_1 \cap G_2 \in \mathcal{H}$ since $\mathcal{H}$ is a filter. For the same reason if only one of them belongs to $\mathcal{H}$, then $G_1 \cap G_2 \notin \mathcal{H}$. The reader can check that for each of these and the remaining cases we have that $\mathbb{I}_{\mathcal{H}}(G_1 \cup G_2) + \mathbb{I}_{\mathcal{H}}(G_1 \cap G_2) = \mathbb{I}_{\mathcal{H}}(G_1) + \mathbb{I}_{\mathcal{H}}(G_2)$. If $\mathcal{G}^* \subseteq \mathcal{G}$ is a directed family and $\mathbb{I}_{\mathcal{H}}(\bigcup^{\uparrow} \mathcal{G}^*) = 1$, then (since $\mathcal{H}$ is completely prime) there exists m such that $m \in \mathcal{H} \cap \mathcal{G}^*$. Then $\mathbb{I}_{\mathcal{H}}(\bigcup^{\uparrow} \mathcal{G}^*) = \mathbb{I}_{\mathcal{H}}(m) = \sup_{G\in\mathcal{G}^*} \mathbb{I}_{\mathcal{H}}(G)$. Therefore $\mathbb{I}_{\mathcal{H}}$ is a continuous $\{0,1\}$-valuation on X. Conversely, if ν is a continuous $\{0,1\}$-valuation on X, it is not difficult to see that $\mathcal{H}_\nu := \{G \in \mathcal{G} \mid \nu(G) = 1\} \in pt(\mathcal{G})$. Clearly $\mathcal{H} = \mathcal{H}_{\mathbb{I}_{\mathcal{H}}}$ and $\nu = \mathbb{I}_{\mathcal{H}_\nu}$. ■

The relation between prime filters, ultrafilters and zero-one measures in lattices has been the theme of intense study. Unfortunately the scope of the present work does not allow us to survey this area. For the interested reader a good starting point in the subject is the work of Bachman and Cohen [BC73][Coh76].

We use the following facts about sober spaces.

Proposition 2.33 *Let $(X, \mathcal{G})$ be a topological space. Let $\mathcal{G}_{pt} := \{\mathcal{O}(G)\}_{G\in\mathcal{G}}$ and $X_s := (pt(\mathcal{G}), \mathcal{G}_{pt})$.*

1. *The map $\eta_X : X \to X_s$ is open and continuous.*
2. *X_s is a sober space and the lattices $(\mathcal{G}, \subseteq)$ and $(\mathcal{G}_{pt}, \subseteq)$ are order isomorphic. If X is T_0, then η_X is injective.*
3. *If X is sober (hence T_0), then η_X is a homeomorphism.*
4. *[Kir93, Lemma 2.2] X is sober if and only if the only continuous $\{0, 1\}$-valuations on X are of the form δ_x for some $x \in X$.*

Proof For parts (1),(2) and (3) see sections 7.1.4 and 7.2.1 in [AJ94] or Section 8.7 in [Hec90]. For part (4) notice that from the previous parts it is clear that X is sober if and only if for all $H \in pt(\mathcal{G})$ there exists $x \in X$ such that $H = \mathcal{N}(x)$; the last equation is equivalent to $\mathbb{I}_H = \delta_x$. By the previous lemma we conclude that X is sober if and only if for all continuous valuations ν there exists $x \in X$ such that $\nu = \mathbb{I}_{H_\nu} = \delta_x$. ■

So Proposition 2.33 gives us an alternative characterisation of sober spaces which many authors use as the definition: a space is sober if and only if every completely prime filter of open sets is the filter of open nbds of a unique point. If X is T_0, then $X_s \equiv (pt(\mathcal{G}), \mathcal{G}_{pt})$ is called the **sobrification** of X. The sobrification can be thought as a kind of completion: since the lattices $(\mathcal{G}, \subseteq)$ and $(\mathcal{G}_{pt}, \subseteq)$ are isomorphic, the sobrification adds the necessary points to make a space sober but respects the structure of the lattice of open sets. Obtaining the sobrification can be thought as a idempotent operation on topological spaces. Part (4) of Proposition 2.33 is due to Kirch, who also showed that a space is sober if and only if the only continuous valuations taking finitely many values are exactly the simple valuations [Kir93, Lemma 2.3][Tix95, Satz 2.2]. Of course the most important property of sober spaces is given by the Hofmann-Mislove theorem [HM81, Theorem 2.16]: the pavings $\mathcal{N}(Q)$ of open nbds of compact saturated sets $Q \in \mathcal{Q}$ are precisely the Scott-open filters in $(\mathcal{G}, \subseteq)$. Now we are able to formulate a negative theorem on extension of valuations for non-sober spaces.

Theorem 2.34 *Let $(X, \mathcal{G})$ be a T_0 and non-sober topological space. If its sobrification is first countable, then there exists a continuous $\{0,1\}$-valuation on X that does not extend to a Borel measure.*

The idea behind the proof is quite simple. Since the space is non-sober then the sobrification has at least one point more, say y. We show that δ_y determines a continuous valuation on X which cannot be extended to a measure; the mass would have to be precisely in the point that is missing.

Proof Since X is non-sober, then there exists $y \in pt(\mathcal{G})$ which is not the filter of open nbds of a point in X. By the construction in the proof of Lemma 2.32 we have that (the indicator function) $\mathbb{I}_y$ is a continuous $\{0,1\}$-valuation on X. Since the function η_X is continuous, then $\nu := \mathbb{I}_y \circ \eta_X^{-1}$ is a continuous $\{0,1\}$-valuation on $pt(\mathcal{G})$. Recall that all the open sets in $pt(\mathcal{G})$ are of the form $\mathcal{O}(G)$ for some $G \in \mathcal{G}$, hence by unwinding the definitions we see that

$$\begin{aligned}
\nu(\mathcal{O}(G)) &:= \mathbb{I}_y(\eta_X^{-1}\mathcal{O}(G)) \\
&= \mathbb{I}_y(\eta_X^{-1}\{H \in pt(\mathcal{G}) \mid G \in H\}) \\
&= \mathbb{I}_y(\{x \in X \mid G \in \eta_X(x)\} \\
&= \mathbb{I}_y(G) \\
&= \delta_y(\mathcal{O}(G)) \qquad\qquad \text{for all } G \in \mathcal{G};
\end{aligned}$$

that is, $\nu = \delta_y$. Since $pt(\mathcal{G})$ is first countable by assumption, then y has a countable fundamental system of open neighborhoods $\{\mathcal{O}(G_n)\}_{n\in\mathbb{N}}$. Let $F := cl_{\mathcal{G}_{pt}}(\{y\})$. For each $n \in \mathbb{N}$ define $E_n := \bigcap_{k=0}^n G_k \cap \eta_X^{-1}F$, then (E_n) is a decreasing sequence of Borel subsets of X. Suppose that $\mathbb{I}_y$ extends to a probability measure $\overline{\mathbb{I}}_y$. Then,

$$\begin{aligned}
\overline{\mathbb{I}}_y(\textstyle\bigcap_{k=0}^n G_k) &= \mathbb{I}_y(\textstyle\bigcap_{k=0}^n \eta_X^{-1}\mathcal{O}(G_k)) \\
&= \mathbb{I}_y(\eta_X^{-1} \textstyle\bigcap_{k=0}^n \mathcal{O}(G_k)) \\
&= \nu(\textstyle\bigcap_{k=0}^n \mathcal{O}(G_k)) \\
&= \delta_y(\textstyle\bigcap_{k=0}^n \mathcal{O}(G_k)) \\
&= 1.
\end{aligned}$$

Since $\mathbb{I}_y(\eta_X^{-1}F^c) = \nu(F^c) = \delta_y(F^c) = 0$, then $\overline{\mathbb{I}}_y(\eta_X^{-1}F) = 1$. It follows that $\overline{\mathbb{I}}_y(E_n) = 1$ for all $n \in \mathbb{N}$. Since $\overline{\mathbb{I}}_y$ is a measure and the sequence (E_n) is decreasing, then $\overline{\mathbb{I}}_y(\bigcap_{n\in\mathbb{N}} E_n) = \inf_{n\in\mathbb{N}} \overline{\mathbb{I}}_y(E_n) = 1$. But the sobrification is a T_0 space and $\{\mathcal{O}(G_n)\}_{n\in\mathbb{N}}$ is a fsn for y; therefore,

$$
\begin{aligned}
\bigcap_{n\in\mathbb{N}} E_n &= \bigcap_{n\in\mathbb{N}} G_n \cap \eta_X^{-1} F \\
&= \eta_X^{-1}(\bigcap_{n\in\mathbb{N}} \mathcal{O}(G_n) \cap cl(y)) \\
&= \eta_X^{-1}(\{y\}) \\
&= \emptyset \qquad\qquad \text{since we choose } y \notin \eta_X(X).
\end{aligned}
$$

Contradicting the fact that $\overline{\mathbb{I}}_y$ is a measure (see Proposition 1.14). Therefore, $\mathbb{I}_y$ is a continuous $\{0,1\}$-valuation on X that cannot be extended to a measure. ■

Next we show a particular instance of a dcpo where the previous theorem holds [AMESD00, Section 6]. The following space was defined by Johnstone [Joh82, Chapter II, 1.9] as an example of a dcpo with a non-sober Scott topology. Let $X := \mathbb{N} \times (\mathbb{N} \cup \{\infty\})$ with $(j,k) \sqsubseteq (m,n)$ if and only if either $j = m$ and $k \leq n$ or $n = \infty$ and $k \leq m$. To see that $(X, \sqsubseteq)$ is a dcpo, by Proposition 1.2, it suffices to check that all the chains in X have a least upper bound. Notice that all the chains in X are either eventually constant or of the form $\{(m, x_n) \mid m \in \mathbb{N},\ (x_n) \text{ non-decreasing infinite sequence in } \mathbb{N}\}$; the first case is trivial, in the second case the lub is given by (m, ∞). Define the function $\nu : \mathcal{G}_\sigma X \to \{0,1\}$ by

$$
\nu(G) := \begin{cases} 1 & \text{if } G \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}
$$

Notice that for all $(j,k) \in X$ with $k \neq \infty$ we have that $\uparrow(j,k) = \{(j,n) \mid k \leq n\} \cup \{(m,\infty) \mid k \leq m\}$. Since (Scott) open sets are upper and the sets of the form $\{(x_j, \infty) \mid x_j \in \mathbb{N} \text{ for } j = 1,...,n\}$ are not open, then every nonempty open set contains all but a finite number of points of the form (m, ∞). Therefore the intersection of any pair of nonempty open sets in X is again nonempty which implies that ν is a modular function and that X is a closed irreducible set (which is obviously not the closure of a unique point). Strictness, monotonicity and continuity are easily verified; hence ν is a continuous $\{0,1\}$-valuation. Nevertheless, by part (1) of Proposition 1.14 ν cannot be extended to a Borel measure. Notice that $F_n := \bigcup_{i=0}^{n} \downarrow(i, \infty)$ is an increasing sequence of closed sets and for all $(j,k) \in X$ we have that $(j,k) \in F_j$; therefore $G_n := F_n^{\mathsf{c}}$ is a decreasing sequence of open sets with $\bigcap_{n\in\mathbb{N}} G_n = \emptyset$, but $\lim_n \nu(G_n) = 1$. By Theorem 2.16, ν also gives an example of a continuous valuation on a dcpo that is not the supremum of a directed family of simple valuations.

3. Extension of valuations on regular and locally compact sober spaces

3.1 Summary

Let $(X, \mathcal{G})$ denote a topological space. Suppose that X is Hausdorff regular (T_3) or locally compact and sober. We show that every locally finite continuous valuation on X extends uniquely to a Borel measure (Theorem 3.23 and Theorem 3.27). As corollaries we obtain the extension result for metric spaces (known) and the patch space of a locally compact sober space (Corollary 3.26). This gives a different proof for the extension result on continuous dcpo's (Corollary 2.22), but the rest of the results are not comparable with those of the previous chapter.

In the previous chapter we presented an extension result for continuous valuations on dcpo's; yet we observed that there exist examples of finite continuous valuations on dcpo's that do not extend to measures. The motivation of this chapter is to find spaces where all locally finite continuous valuations extend uniquely to Borel measures. The historic note (in the context of Domain Theory) is as follows. The extension result for T_3 spaces is a straightforward application of the work on the construction of measures by Topsøe [Top70a][Top70b][Top78]. Norberg [Nor89] established the result for σ-finite continuous valuations on second countable locally compact sober spaces; Lawson [Law82] for finite valuations on these last spaces and stably compact spaces, which implies the extension for compact pospaces (as defined by Nachbin [Nac65]). Norberg and Vervaat [NV97] gave an extension theorem for capacities which implies the extension result for locally finite valuations on stably compact spaces. It is important to notice that, although stably compact spaces are in particular locally compact sober spaces, the results presented here do *not* imply those in [Law82] and [NV97]. This is because in the latter results the valuations extend to measures on σ-algebras that are larger than the Borel σ-algebra of the topological space (see Section 3.6 for a detailed account). The question whether on a sober space every finite continuous valuation extends to a Borel measure remains open. We conjecture that it is not true, since sober spaces are a very broad class of spaces which includes Hausdorff spaces, but we have not been able to construct a counterexample.

The chapter is organized as follows. Section 3.2 reviews the theory of compact monocompact approximating pavings for measures. These results are also used in the next chapter for talking about disintegration and regular conditional probability distributions. Section 3.3 encapsulates some basic facts about sober, coherent and locally compact sober spaces. Section 3.4 introduces the space of lenses, and shows it is a monocompact paving and a dcpo. These results are used in later chapters for constructing maximal point space representations of the patch space (Section 5.5). Section 3.5 contains the main extension results. Section 3.6 compares our results with those of Norberg, Vervaat and Lawson. We also present another proof of Lawson's extension result for stably locally compact spaces which does not use the Riesz representation theorem. Finally, Section 3.7 discuses some properties of the patch space as a bitopological space.

3.2 A review of compact and monocompact approximating pavings

In this section we review the main properties of compact and monocompact pavings and tailor some results to the form we need in the following section. One of the central notions in probability and measure theory is that of a compact paving. A set $\mathcal{C}$ is said to be a [semi] **compact** paving on Ω if for every [countable] family $\{K_j\}_{j\in J}$ of sets in $\mathcal{C}$ such that $\bigcap_{j\in J} K_j = \emptyset$, there exists a finite subfamily $\{K_{j_k}\}_{k=1}^{n}$ such that $\bigcap_{k=1}^{n} K_{j_k} = \emptyset$. If $\mathcal{C}$ is a semicompact paving on Ω, then $\mathcal{C} \cup \{\Omega\} \cup 2^{\Omega}_{f}$ is a semicompact paving. Any subset of a semicompact paving is clearly a semicompact paving. Suppose that $\mathcal{C}$ is a semicompact paving, let $\overline{\mathcal{C}}$ be the paving constructed by taking countable intersections of finite unions of elements of $\mathcal{C} \cup \{\emptyset, \Omega\}$. Then $\overline{\mathcal{C}}$ is the minimum $(\cup f, \cap c, \emptyset, \Omega)$-stable semicompact paving $\overline{\mathcal{C}}$ containing $\mathcal{C}$ [HJ94, Section 1.29][Top78, p. 359]. That is why in the literature semicompact pavings are also called semicompact lattices.

Definition 3.1 *(as in [Top79]) Let $\mathcal{P}$ be a paving on Ω and $\mu : \mathcal{P} \to \mathbb{R}_+$ be a finite set function. We say that a paving $\mathcal{C}$ on Ω is an* ***approximating paving*** *for μ if for all $E_1 \in \mathcal{P}$ and $\varepsilon > 0$ there exist $E_2 \in \mathcal{P}$ and $K \in \mathcal{C}$ such that*

$E_2 \subseteq K \subseteq E_1$ *and* $\mu(E_1) - \mu(E_2) \leq \varepsilon$.

Let $\mu : \mathcal{A} \to \mathbb{R}_+$ be a content on an algebra $\mathcal{A}$ over a set Ω. We say that μ is a **compact content** [Mar53, Section 4] if there exists a semicompact paving $\mathcal{C}$ that approximates μ on Ω. If $\mathcal{A}$ is a σ-algebra we say that μ is a **compact measure**. When we say that μ is a $\mathcal{C}$-compact measure (or content), we mean that $\mathcal{C}$ is a semicompact approximating paving for μ. We have the following result,

Proposition 3.2 *(Marczewski) Let (Ω, Γ) be a measurable space and $\mathcal{A}$ be an algebra on Ω.*

1. *If $\mu : \mathcal{A} \to \mathbb{R}_+$ is a compact content, then it can be extended uniquely to a measure on $\sigma(\mathcal{A})$.*
2. *$\mu : \Gamma \to \overline{\mathbb{R}}_+$ is a compact measure if and only if there exists an $(\cup f, \cap c)$-semicompact paving $\mathcal{C} \subseteq \Gamma$ such that*

$$\mu(F) = sup\{\mu(K) \mid K \in \mathcal{C} \text{ and } K \subseteq F\} \text{ for all } F \in \Gamma.$$

Proof See Section 4 in [Mar53]. ■

Compact measures play a central role in measure and probability theory since the existence of regular conditional probability distributions, in general, can only be ensured for compact measures (see Section 4.4). They also provide the basis for the proof of Kolmogorov's Theorem (Proposition 1.20) in its most general form (Proposition 4.6). We already observed that if a paving $\mathcal{P}$ is $(\cap f)$-stable, then any intersection can be rewritten as a filtered intersection. We might ask ourselves if a $(\cap_\downarrow)$-stable paving $\mathcal{C}$ not containing the empty set, is compact? The answer is no in general, but obviously yes when $\mathcal{C}$ is already $(\cap f)$-stable (see [Top78, Section 6] for discussion). Examples of $(\cap_\downarrow)$-stable pavings that are not always closed under finite intersections are provided by the set of nonempty compact saturated subsets $\mathcal{Q}\backslash\{\emptyset\}$ of a sober topological space $(X, \mathcal{G})$ [HM81, Proposition 2.19, Theorem 4.8]. Continuous dcpo's are, of course, important cases of this class of spaces. This motivates the following definition which was firstly introduced by Mallory [Mal75] and later generalised by Topsøe [Top78, Section 6][Top79].

Definition 3.3 *[Top79] A set $\mathcal{C}$ is a* ***monocompact*** *paving on Ω if for every decreasing sequence $\{K_n\}_{n\in\mathbb{N}}$ of nonempty sets in $\mathcal{C}$ we have that $\bigcap_\downarrow K_n \neq \emptyset$.*

As we mentioned before, the compact saturated subsets of a sober topological space are an example of a monocompact paving which is not necessarily compact. In fact, if a topological space $(X, \mathcal{G})$ is T_1 and sober then $\mathcal{K}X = \mathcal{Q}X$, and from part (3) of Proposition 3.9 below if follows that $\mathcal{K}X$ is a monocompact paving. The set of nonempty compact subsets of a Hausdorff topological space is obviously a compact paving.

Definition 3.4 *[Top78, Section 6] Let $\mathcal{P}$ be a $(\emptyset, \cap f)$-paving on a set Ω. Let $\mu : \mathcal{P} \to \mathbb{R}_+$ be a finite set function. We say that μ is **tight** if*

$\mu_*(K_2 \backslash K_1) = \mu(K_2) - \mu(K_1)$ for all $K_1, K_2 \in \mathcal{P}$ with $K_1 \subseteq K_2$.

Recall from Section 1.7 that

$$\begin{aligned} \mu_*(K_2 \backslash K_1) \quad &= \sup\{\textstyle\sum_{j=1}^{n} \mu(E_j) \mid \; E_1, \ldots, E_n \text{ disjoint } \in \mathcal{P} \text{ and} \\ &\qquad\qquad\qquad\qquad \textstyle\sum_{j=1}^{n} E_j \subseteq K_2 \backslash K_1\}. \end{aligned}$$

Proposition 3.5 *(Topsøe) Let $\mathcal{P}$ be a $(\emptyset, \cap f)$-paving (that is, a π-system) on a set Ω. Let $\mu : \mathcal{P} \to \mathbb{R}_+$ be a tight set function. If there exists a monocompact paving approximating μ on $\mathcal{P}$, then μ can be extended uniquely to a measure on $\sigma(\mathcal{P})$.*

Proof See Theorem 6.1 in [Top78] and Lemma 1 in [Top79]. ■

Notice that for a compact measure μ we could find an approximating semi-compact paving contained in the σ-algebra where μ was defined. For mono-compact measures this cannot be ensured. As a consequence of Proposition 1.23 we immediately get,

Corollary 3.6 *Let $(X, \mathcal{G})$ be a topological space, $\nu : \mathcal{G} \to \mathbb{R}_+$ be a finite valuation and $\overline{\nu}$ its unique extension to a content on $\mathcal{A}(\mathcal{G})$. If there exists a monocompact paving $\mathcal{C}$ approximating $\overline{\nu}$ on $Cres(X)$, then ν can be extended uniquely to a Borel measure.*

Proof Notice that $Cres(X)$ is a $(\emptyset, \cap f)$-paving on X and $\overline{\nu} : Cres(X) \to \mathbb{R}_+$ is a set function. It suffices to show that $\overline{\nu}$ is tight. A note of caution should be issued here, since the inner measure $\overline{\nu}_*$ must be taken with respect to the paving $Cres(X)$ not with respect to the algebra $\mathcal{A}(\mathcal{G})$. Let $C_1, C_2 \in Cres(X)$ with $C_1 \subseteq C_2$. Since $\mathcal{A}(\mathcal{G})$ is the algebra generated by the semialgebra $Cres(X)$ and $C_2 \backslash C_1 \in \mathcal{A}(\mathcal{G})$, then $C_2 \backslash C_1 = \sum_{j=1}^{n} E_n$ with $E_n \in Cres(X)$ for $j = 1,...,n$. Since $\overline{\nu}$ is a content on $\mathcal{A}(\mathcal{G})$ we have that

$$\begin{aligned} \overline{\nu}_*(C_2 \backslash C_1) \quad &\geq \textstyle\sum_{j=1}^{n} \overline{\nu}(E_n) \qquad \text{by definition of inner measure} \\ &= \overline{\nu}(\textstyle\sum_{j=1}^{n} E_n) \\ &= \overline{\nu}(C_2 \backslash C_1) \\ &= \overline{\nu}(C_2) - \overline{\nu}(C_1) \end{aligned}$$

But $\overline{\nu}_*(C_2 \backslash C_1) \leq \overline{\nu}(C_2 \backslash C_1)$, therefore $\overline{\nu}$ is tight. It follows from Proposition 3.5 that $\overline{\nu}$ can be extended to a measure on $\sigma(Cres(X)) = \mathcal{B}X$. ■

Remark 3.7 *This result also follows from Theorem 1.2 in [Mal75]. Compare with Theorems 2.1 and 4.1 in [Pet51].*

Problem 3.8 *Characterise the topological spaces for which $\mathcal{K}$ or $\mathcal{Q}$ are semicompact or monocompact pavings.*

3.3 Auxiliary properties of sober and coherent spaces

The following facts about sober and coherent spaces will be used frequently.

Proposition 3.9 *Let $(X, \mathcal{G})$ be a sober space and $\sqsubseteq_s$ the specialisation order. Then,*

1. *$(X, \sqsubseteq_s)$ is a dcpo and $\mathcal{G}_u \subseteq \mathcal{G} \subseteq \mathcal{G}_\sigma$. That is, $(X, \mathcal{G})$ is a monotone convergence space and $\mathcal{G}$ is order consistent.*
2. *$\mathcal{Q}$ is $(\cap_\downarrow)$-stable.*
3. *If $\{Q_i\}_{i \in I}$ is a filtered family of sets in $\mathcal{Q}$ and $G \in \mathcal{G}$ with $\bigcap_\downarrow Q_i \subseteq G$, then there exists $j \in I$ such that $Q_j \subseteq G$.*
4. *$(\mathcal{Q} \backslash \{\emptyset\}, \supseteq)$ is a dcpo and $int(Q_1) \supseteq Q_2$ implies that $Q_1 \ll Q_2$.*

Proof For part (1) see Propositions 7.2.12 and 7.2.13 in [AJ94] or Exercises II-3.16 and II-3.17 in [GHK$^+$80]. For parts (2) and (3) see Proposition 2.19 in [HM81] or Corollary 7.2.11 in [AJ94]. For part (4) see Lemma 2.21 in [HM81]. ■

Proposition 3.10 *If $(X, \mathcal{G})$ is locally compact and sober, then*

1. *if $Q \in \mathcal{Q}$ and $Q \subseteq G \in \mathcal{G}$, then there exists $Q' \in \mathcal{Q}$ such that $Q \subseteq int(Q') \subseteq Q' \subseteq G$.*
2. *$(\mathcal{Q} \backslash \{\emptyset\}, \supseteq)$ is a continuous dcpo where $Q_1 \ll Q_2$ if and only if $int(Q_1) \supseteq Q_2$.*
3. *$(\mathcal{F}, \supseteq)$ is a continuous lattice where $F_1 \ll F_2$ if and only if there exists $Q \in \mathcal{Q}$ such that $F_1 \supseteq Q^{\mathsf{c}} \supseteq F_2$.*
4. *The patch topology $\mathcal{G}_p$ is $T_{3\frac{1}{2}}$ (therefore Hausdorff).*
5. *If X is second countable, then $(X, \mathcal{G}_p)$ is a Polish space.*
6. *If X is stably locally compact, then $(X, \mathcal{G}_p)$ is a locally compact Hausdorff space.*

Proof (1) Since X is locally compact, then for every $y \in Q$ there exists $Q_y \in \mathcal{Q}$ such that $y \in int(Q_y) \subseteq Q_y \subseteq G$. Then $\{int(Q_y)\}_{y \in Q}$ is an open cover for Q. Since Q is compact, we can extract a finite subcovering $\{int(Q_{y_k})\}_{k=1}^n$. Let $G' := \bigcup_{k=1}^n int(Q_{y_k})$, and $Q' := \bigcup_{k=1}^n Q_{y_k}$. Note that

$Q' \in \mathcal{Q}$ since $\mathcal{Q}$ is $(\cup f)$-stable. Therefore $Q \subseteq G' \subseteq Q' \subseteq G$. For part (2) see Proposition 2.19 in [HM81]. Part (3) follows from parts (1) and (2) (see Exercise III-1.13 in [GHK$^+$80]). For parts (4), (5) and (6) see Corollary 4.6 and part (6) of Theorem 4.8 in [HM81]. ■

Lemma 3.11 *Let $(X, \mathcal{G})$ be a second countable locally compact sober space. Then,*

1. *$(X, \mathcal{G}_{cc})$ and $(X, \mathcal{G}_p)$ are second countable and $\mathcal{B}X = \mathcal{B}_p X$.*
2. *The dcpo's $(\mathcal{Q}\backslash\{\emptyset\}, \supseteq)$ and $(\mathcal{F}, \supseteq)$ are ω-continuous.*

Proof (Similar argument as in Lemma 2.9) Let $\mathcal{J} = \{J_n\}_{n\in\mathbb{N}}$ be a countable base for $\mathcal{G}$. Without loss of generality, we assume that $\mathcal{J}$ is $(\cup f)$-stable. Let $I = \{(h,k) \in \mathbb{N}^2 \mid \exists Q \in \mathcal{Q} \text{ with } J_h \subseteq Q \subseteq J_k\}$. Use a choice function to construct a collection $\mathcal{H} = \{Q^{\mathsf{c}}_{(h,k)} \mid (h,k) \in I\}$ such that $J_h \subseteq Q_{(h,k)} \subseteq J_k$ for all $(h,k) \in I$. The set $\mathcal{H}$ is countable since $I \subseteq \mathbb{N}^2$.

1. **Claim:** If $Q \in \mathcal{Q}$ and $Q \subseteq G \in \mathcal{G}$, then there exists $J_n \in \mathcal{J}$ with $Q \subseteq J_n \subseteq G$. Since $\mathcal{J}$ is a base for the topology, then $G = \bigcup_{n\in\mathcal{C}} J_n$. But Q is compact, therefore there exists a finite $\mathcal{C}_0 \subseteq_f \mathcal{C}$ such that $Q \subseteq \bigcup_{n\in\mathcal{C}_0} J_n \subseteq G$. Since $\mathcal{J}$ is $(\cup f)$-stable we have that $\bigcup_{n\in\mathcal{C}_0} J_n \in \mathcal{J}$.
2. **Claim:** If $Q \in \mathcal{Q}$ and $x \notin Q$, then there exists $J_n \in \mathcal{J}$ such that $Q \subseteq J_n$ and $x \notin J_n$. Since $Q = sat(Q)$, then Q is the intersection of its open neighborhoods. Hence there exist $G \in \mathcal{G}$ with $Q \subseteq G$ and $x \notin G$. The claim follows from the previous claim.
3. **Claim:** $\mathcal{Q} \subseteq \mathcal{B}X$. If $Q \in \mathcal{Q}$, then by claim (2) Q can be expressed as the countable intersection $\bigcap_{m\in\mathbb{N}}\{J_m \mid Q \subseteq J_m\}$. Since $\mathcal{J} \subseteq \mathcal{G}$ then Q is a Borel set.
4. **Claim:** $\mathcal{H}$ is a countable base for $\mathcal{G}_{cc}$. Recall that $\{Q^{\mathsf{c}} \mid Q \in \mathcal{Q}\}$ is a base for the cocompact topology. Suppose that $x \in Q^{\mathsf{c}}$ and $Q \in \mathcal{Q}$. Then by claim (2) there exists $J_k \in \mathcal{J}$ such that $Q \subseteq J_k$ and $x \notin J_k$. By part (1) of Proposition 3.10 we can find $Q' \in \mathcal{Q}$ with $Q \subseteq int(Q') \subseteq Q' \subseteq J_k$. By claim (1) applied to Q and $int(Q')$ there exists $J_h \in \mathcal{J}$ such that

$$Q \subseteq J_h \subseteq Q' \subseteq J_k \text{ and } x \notin J_k.$$

It follows that $(h,k) \in I$, and therefore

$$x \in J_k^{\mathsf{c}} \subseteq Q^{\mathsf{c}}_{(h,k)} \subseteq J_h^{\mathsf{c}} \subseteq Q^{\mathsf{c}}.$$

Hence $Q^c = \bigcup\{H \in \mathcal{H} \mid H \subseteq Q^c\}$.

$\boxed{1}$ Claim (4) shows that $(X, \mathcal{G}_{cc})$ is second countable. Recall from Section 1.4 that $\{G \cap Q^c \mid G \in \mathcal{G}, Q \in \mathcal{Q}\}$ is a base for $(X, \mathcal{G}_p)$. It follows immediately that $\mathcal{JH} = \{J \cap H \mid J \in \mathcal{J}$ and $H \in \mathcal{H}\}$ is a countable base for $(X, \mathcal{G}_p)$. By claim (3) we have that $\mathcal{JH} \subseteq \mathcal{B}X$. Therefore $\mathcal{G}_p \subseteq \mathcal{B}X$, hence $\mathcal{B}_p X = \sigma(\mathcal{G}_p) \subseteq \mathcal{B}X$. This concludes the first part.

$\boxed{2}$ Suppose that $Q_1 \ll Q_2$ with $Q_i \in \mathcal{Q}\backslash\{\emptyset\}$. Then by Proposition 1.3 there exists $Q \in \mathcal{Q}$ such that $Q_1 \ll Q \ll Q_2$. By part (2) of Proposition 3.10 we have that

$$Q_1 \supseteq int(Q_1) \supseteq Q \supseteq int(Q) \supseteq Q_2.$$

By claim (1) above it follows that there exist J_{m_1} and J_{m_2} in $\mathcal{J}$ such that:

$$Q_1 \supseteq J_{m_1} \supseteq Q \supseteq J_{m_2} \supseteq Q_2.$$

It follows that $(m_1, m_2) \in I$. Therefore $Q_1 \ll Q_{(m_1,m_2)} \ll Q_2$. Again by Proposition 1.3 we conclude that $\{Q_{(h,k)} \mid (h,k) \in I\}$ is a countable basis for $\mathcal{Q}\backslash\{\emptyset\}$. Therefore $(\mathcal{Q}\backslash\{\emptyset\}, \dot{\supseteq})$ is ω-continuous. Finally consider the set $I' = \{(h,k) \in I^2 \mid \exists F \in \mathcal{F}$ with $Q_h^c \supseteq F \supseteq Q_k^c\}$. Use a choice function to construct a set $\{F_{(h,k)} \mid (h,k) \in I'\}$ such that $Q_h^c \supseteq F_{(h,k)} \supseteq Q_k^c$ for all $(h,k) \in I'$. By similar interpolation arguments as above it is easy to see that the set constructed is a countable basis for $(\mathcal{F}, \supseteq)$. ■

Lemma 3.12 *If $(X, \mathcal{G})$ is coherent, then*

1. *$\mathcal{Q}$ is $(\cap)$-stable.*
2. *$\mathcal{Q} \subseteq \mathcal{K}_p$. That is, every compact saturated set is compact in the patch topology.*
3. *$(X, \mathcal{G}_p)$ is compact if and only if $X \in \mathcal{Q}$.*

Proof

1. Since $\mathcal{Q}$ is $(\cap f)$-stable, then any arbitrary intersection can be rewritten as a filtered intersection. The conclusion follows from part (2) of Proposition 3.9.
2. The argument is similar to the one in the proof of Theorem III-1.9 in [GHK+80]. Suppose that $P \in \mathcal{Q}$. By Alexander Lemma (Lemma 1.6) it suffices to show that every open cover of patch *subbasic* opens for P has a finite subcovering. Suppose that $P \subseteq \bigcup_{i \in I} Q_i^c \cup \bigcup_{j \in J} G_j$ where $Q_i \in \mathcal{Q}$ and $G_j \in \mathcal{G}$ for all $i \in I$ and $j \in J$. Let $Q = (\bigcup_{i \in I} Q_i^c)^c =$

$\bigcap_{i\in I} Q_i$. Then $P\cap Q \subseteq \bigcup_{j\in J} G_j$ and by part (1) we know that $P\cap Q \in \mathcal{Q}$. So there exists a finite $J_f \subseteq J$ such that $P\cap Q \subseteq \bigcup_{j\in J_f} G_j = G$. Again, since G is open and $P\cap Q \subseteq G$, by part (3) of Proposition 3.9 there exists a finite $I_f \subseteq I$ such that $P\cap\bigcap_{i\in I_f} Q_i \subseteq G$. Hence $P\cap G^{\mathsf{c}} \subseteq \bigcup_{i\in I_f} Q_i^{\mathsf{c}}$. Since $X = G\cup G^{\mathsf{c}}$ it follows that $\bigcup_{i\in I_f} Q_i^{\mathsf{c}} \cup \bigcup_{j\in J_f} G_j$ is a finite subcovering for P.

3. If $X \notin \mathcal{Q}$, then there exists an open cover $\mathcal{C} \subseteq \mathcal{G}$ for X that does not admit a finite subcovering. But $\mathcal{C}$ is also a open cover for X in the patch topology. Therefore X cannot be compact. Conversely if $X \in \mathcal{Q}$, then by part (2) we have that $X \in \mathcal{K}_p$. ■

Regarding part (3), quite often the phrase 'on a coherent space the patch topology is compact' is found in the literature. If we define coherence as sobriety plus $\mathcal{Q}$ is closed under finite intersections, then this is correct provided that we implicitly allow the empty intersection, which means $X \in \mathcal{Q}$. This consideration is often forgotten by some authors.

Definition 3.13 *Let $(X,\mathcal{G})$ be any topological space. The **Fell topology** [Fel62] $\mathcal{G}_F$ on the paving $\mathcal{F}$ of closed subsets of X is defined as the topology generated by the families*

- $\{F\in\mathcal{F} \mid F\cap K = \emptyset\}_{K\in\mathcal{K}}$
- $\{F\in\mathcal{F} \mid F\cap G \neq \emptyset\}_{G\in\mathcal{G}}$

Proposition 3.14 *Suppose that $(X,\mathcal{G})$ is any topological space. Then*

1. *$(\mathcal{F},\mathcal{G}_F)$ is compact.*
2. *If X is locally compact, then*
   - *$(\mathcal{F},\mathcal{G}_F)$ is compact Hausdorff.*
   - *$\{F\in\mathcal{F} \mid F\cap K = \emptyset\}_{K\in\mathcal{K}}$ is a base for the Scott topology $\mathcal{G}_\sigma(\mathcal{F})$ on $(\mathcal{F},\supseteq)$.*
   - *$\{F\in\mathcal{F} \mid F\cap G \neq \emptyset\}_{G\in\mathcal{G}}$ is a subbase for the lower topology $\mathcal{G}_l(\mathcal{F})$ on $(\mathcal{F},\supseteq)$.*
   - *$\mathcal{G}_F$ agrees with the Lawson topology $\mathcal{G}_\lambda(\mathcal{F})$ on $(\mathcal{F},\supseteq)$.*

Proof See Lemma 1 and Theorem 1 in [Fel62]; and Exercise III-1.13 in [GHK+80]. ■

3.4 Spaces of lenses

Definition 3.15 *Let $(X, \mathcal{G})$ be any topological space. The* ***space of lenses*** *$Lens(X)$ is defined as*

$$Lens(X) = \{Q \cap F \mid Q \cap F \neq \emptyset, Q \in \mathcal{Q}, F \in \mathcal{F}\}$$

Compare this definition with Definition 6.2.16 in [AJ94] and Definition 4.43 in [Mis98]. We have the following facts.

Proposition 3.16 *Let $(X, \mathcal{G})$ be a topological space and L be any member of $Lens(X)$. Then,*

1. *L is convex, i.e. $L = \uparrow L \cap \downarrow L$.*
2. *$L \in \mathcal{K}$ therefore $\uparrow L \in \mathcal{Q}$.*
3. *A canonical representation for L is $\uparrow L \cap cl(L)$.*
4. *L is closed in the patch topology.*
5. *If $(X, \mathcal{G})$ is coherent, then L is compact in the patch topology. That is, $Lens(X) \subseteq \mathcal{K}_p$.*

Proof The proofs of the first four claims are the same as for dcpo's [AJ94, Propositions 6.2.5, 6.2.17 and Corollary 6.2.21]; we include them for completeness. For the rest of the proof assume that $L = Q \cap F$ with $Q \in \mathcal{Q}$ and $F \in \mathcal{F}$.

1. $L \subseteq \uparrow L \cap \downarrow L$ is obvious. If $x \in \uparrow L \cap \downarrow L$, then there exist $q, y \in L$ such that $q \sqsubseteq_s x \sqsubseteq_s y$. Since $q \in Q$ (an upper set) and $y \in F$ (a lower set), then $x \in Q \cap F = L$.
2. Since the intersection of a closed set with a compact set is always compact [Bou66, Proposition I-(9.3)-3], we have that $L \in \mathcal{K}$ and the claim follows.
3. $L \subseteq \uparrow L \cap cl(L)$ is obvious. Since $L \subseteq Q$, then $\uparrow L \subseteq \uparrow Q = Q$. In the same way $L \subseteq F$ implies that $cl(L) \subseteq cl(F) = F$. Therefore, $\uparrow L \cap cl(L) \subseteq Q \cap F = L$ and the claim follows.
4. This is obvious since Q^{c} and F^{c} are subbasic open sets of $\mathcal{G}_p$.
5. Notice that $F \in \mathcal{F}_p$ and by part (2) of Lemma 3.12 we know that $Q \in \mathcal{K}_p$. Again the conclusion follows since the intersection of a closed set with a compact set is compact. ■

Next we show a technical lemma.

Lemma 3.17 *Let $(X, \mathcal{G})$ be a sober space. Suppose that $\{Q_i\}_{i\in I}$ and $\{F_j\}_{j\in J}$ are directed families of sets in $(\mathcal{Q}, \supseteq)$ and $(\mathcal{F}, \supseteq)$ respectively. Let $Q = \bigcap_{\downarrow} Q_i$ and $F = \bigcap_{\downarrow} F_j$. If $Q \cap F = \emptyset$, then there exist $h \in I$ and $k \in J$ such that $Q_h \cap F_k = \emptyset$.*

Proof Since $Q \cap F = \emptyset$, then $\bigcap_{\downarrow} Q_i \subseteq F^{\mathsf{c}}$. By part (3) of Proposition 3.9, there must exist an $h \in I$ such that $Q_h \subseteq F^{\mathsf{c}} = \bigcup^{\uparrow} F_j^{\mathsf{c}}$. Since Q_h is compact, then there exists a finite $J_f \subseteq J$ such that $Q_h \subseteq \bigcup^{\uparrow}_{j\in J_f} F_j^{\mathsf{c}}$. By directedness of the family $\{F_j\}_{j\in J}$ we can find $k \in J$ such that $Q_h \subseteq F_k^{\mathsf{c}}$. That is, $Q_h \cap F_k = \emptyset$. ■

Recall that a dcpo $(D, \sqsubseteq)$ is bounded complete if for every $x_1, x_2 \in D$ such that there exists $y \in D$ with $x_1, x_2 \sqsubseteq y$, then $x_1 \vee x_2$ exists. The following proposition extends Lemma 3.4 in [NV97] for non-coherent spaces.

Theorem 3.18 *Suppose that $(X, \mathcal{G})$ is a sober space. Then,*

1. *$(Lens(X), \supseteq)$ is a dcpo and $\{\{x\} \mid x \in X\} = \max(Lens(X))$.*
2. *If X is coherent, then $Lens(X)$ is bounded complete.*

Proof For the first part suppose that $\{L_i\}_{i\in I}$ is a $\supseteq$-directed family of lenses. Notice that if $L_i \supseteq L_j$ then $\uparrow L_i \supseteq \uparrow L_j$ and $cl(L_i) \supseteq cl(L_j)$. It follows that $\{\uparrow L_i\}_{i\in I}$ and $\{cl(L_i)\}_{i\in I}$ are directed families of sets in $(\mathcal{Q}, \supseteq)$ and $(\mathcal{F}, \supseteq)$ respectively. Consider

$$\begin{aligned} L &= \textstyle\bigcap_{\downarrow} L_i \\ &= \textstyle\bigcap_{\downarrow}(\uparrow L_i \cap cl(L_i)) \\ &= (\textstyle\bigcap_{\downarrow} \uparrow L_i) \cap (\textstyle\bigcap_{\downarrow} cl(L_i)). \end{aligned}$$

Clearly $F = \bigcap_{\downarrow} cl(L_i)$ is a closed set and by part (2) of Proposition 3.9 we know that $Q = \bigcap_{\downarrow} \uparrow L_i \in \mathcal{Q}$. So we only need to check that $L \neq \emptyset$. Suppose on the contrary that $L = Q \cap F = \emptyset$. Then by Lemma 3.17 there exist $h, k \in I$ such that $\uparrow L_h \cap cl(L_k) = \emptyset$. Since the family $\{L_i\}_{i\in I}$ is directed, we can find $j \in I$ such that $\uparrow L_h \supseteq \uparrow L_j$ and $cl(L_k) \supseteq cl(L_j)$. This means that $L_j = \emptyset$, a contradiction since lenses are by definition nonempty. The second claim is trivial since $\{x\} = \uparrow x \cap \downarrow x$ is a lens.

For the second part we just have to notice that if $L_1 := Q_1 \cap F_1$ and $L_2 := Q_2 \cap F_2$ are bounded by $L_3 \in Lens(X)$, then $L := L_1 \cap L_2$ is nonempty since $L \supseteq L_3 \neq \emptyset$ and $L \in Lens(X)$ (since $Q_1 \cap Q_2 \in \mathcal{Q}$). Clearly $L = L_1 \vee L_2$. ■

Corollary 3.19 *If $(X, \mathcal{G})$ is a sober space, then $Lens(X)$ is a monocompact* *paving.*

Problem 3.20 *If $(X, \mathcal{G})$ is a sober space, is $(Lens(X), \supseteq)$ a sober space with the Scott topology?*

3.5 The extension results

In this section we prove the main extension results: if a space is T_3 or locally compact and sober, then every locally finite continuous valuation extends uniquely to a Borel measure. As two immediate corollaries we get that every locally finite continuous valuation on a metric space extends uniquely to a measure and the same is true for a valuation that is continuous with respect to the patch topology of a locally compact sober space. We begin with the extension result for T_3 spaces. Since the proof is an easy consequence of the following definition and result we can say that the result is essentially due to Topsøe.

Definition 3.21 *Let Ω be any set and $\mathcal{C}$, $\mathcal{P}$ be pavings on Ω with $\emptyset \in \mathcal{P}$. Let $\lambda : \mathcal{P} \to \mathbb{R}_+$ be a finite and monotone set function. We say that λ is* ***τ-smooth at*** *$\emptyset$ with respect to the paving $\mathcal{C}$, if for all $\mathcal{C}^* \subseteq \mathcal{C}$ with $\mathcal{C}^* \downarrow \emptyset$ we have*

$$\lambda(\emptyset) = 0 = \inf\{\lambda(A) \mid \exists B \in \mathcal{C}^* \textit{ with } B \subseteq A\}$$

Proposition 3.22 *(Topsøe) Let $(X, \mathcal{G})$ be a T_3 topological space and ν be a finite valuation which is τ-smooth at $\emptyset$ with respect to $\mathcal{F}$. Then there exists a largest regular, τ-smooth Borel measure μ such that $\mu(G) \leq \nu(G)$ for all $G \in \mathcal{G}$. This measure is given by the formula*

$$\mu(A) = \sup_{F \subseteq A} \lambda(F) \textit{ for all } A \in \mathcal{B}X$$

where $\lambda(F) := \inf_{G \supseteq F} \nu(G)$ for all $F \in \mathcal{F}$.

Proof See Theorem 3 in [Top70a]. ■

Theorem 3.23 ***Extension Theorem for Regular Hausdorff Spaces***

Let $(X, \mathcal{G})$ be a T_3 topological space and ν be a locally finite continuous valuation on X. Then ν extends uniquely to a regular, τ-smooth Borel measure.

Proof By Theorem 2.13 it suffices to show the result for finite valuations. For simplicity assume that $\nu(X) = 1$. By Proposition 1.23 ν extends uniquely to a content $\overline{\nu}$ on $\mathcal{A}(\mathcal{G})$. For any filtered family $\{F_i\}_{i \in I}$ in $\mathcal{F}$ we have that

$$\begin{aligned}
\overline{\nu}(\textstyle\bigcap_{\downarrow} F_i) &= 1 - \nu(\textstyle\bigcup^{\uparrow} F_i^{\mathsf{c}}) \\
&= 1 - \sup_{i \in I} \nu(F_i^{\mathsf{c}}) \quad \text{since } \nu \text{ is continuous} \\
&= \inf_{i \in I}(1 - \nu(F_i^{\mathsf{c}})) \\
&= \inf_{i \in I} \overline{\nu}(F_i)
\end{aligned}$$

Define $\lambda : \mathcal{F} \to [0,1]$ as $\lambda(F) := \inf_{G \supseteq F} \nu(G)$ for all $F \in \mathcal{F}$. Let G be any element of $\mathcal{G}$. Since X is regular, then for every $x \in G$ there exists $F_x \in \mathcal{F}$ such that $x \in int(F_x)$ and $F_x \subseteq G$. Therefore $G = \bigcup_{x \in G} int(F_x)$. Rewriting this union as a directed union and using the fact that ν is continuous, then for every $\varepsilon > 0$ there must exist $x(1),...,x(n)$ in G such that

$$\begin{aligned}
\nu(G) - \varepsilon &< \nu(\textstyle\bigcup_{i=1}^{n} int(F_{x(i)})) \\
&\leq \overline{\nu}(\textstyle\bigcup_{i=1}^{n} F_{x(i)}) \quad \text{by monotonicity of } \overline{\nu} \\
&\leq \nu(G)
\end{aligned}$$

It follows immediately that $\overline{\nu}(G) = \sup_{F \subseteq G} \overline{\nu}(F)$. Therefore for all $F \in \mathcal{F}$ we have that

$$\begin{aligned}
\overline{\nu}(F) &= 1 - \nu(F^{\mathsf{c}}) \\
&= 1 - \sup\{\overline{\nu}(F_1) \mid F_1 \subseteq F^{\mathsf{c}}\} \\
&= \inf\{1 - \overline{\nu}(F_1) \mid F_1 \subseteq F^{\mathsf{c}}\} \\
&= \inf\{\nu(F_1^{\mathsf{c}}) \mid F \subseteq F_1^{\mathsf{c}}\} \\
&= \lambda(F)
\end{aligned}$$

In order to apply the previous proposition we need to show that ν is τ-smooth at $\emptyset$ with respect to $\mathcal{F}$. Let $\mathcal{C}$ be a filtered subset of $\mathcal{F}$ such that $\mathcal{C}\downarrow\emptyset$. Then

$$\begin{aligned}
\inf\{\nu(G) \mid \exists F \in \mathcal{C} \text{ with } F \subseteq G\} &= \inf\{\lambda(F) \mid F \in \mathcal{C}\} \\
&= \inf\{\overline{\nu}(F) \mid F \in \mathcal{C}\} \\
&= \overline{\nu}(\textstyle\bigcap_{\downarrow} \mathcal{C}) \quad \text{from above} \\
&= \nu(\emptyset) = 0.
\end{aligned}$$

Then by the previous proposition we have that there exists a maximum regular, τ-smooth Borel measure μ given by

$$\mu(A) = \sup_{F \subseteq A} \lambda(F) = \sup_{F \subseteq A} \overline{\nu}(F) \text{ for all } A \in \mathcal{B}X$$

In particular for all $G \in \mathcal{G}$ we have that $\mu(G) = \sup_{F \subseteq G} \overline{\nu}(F) = \nu(G)$. So μ is indeed an extension of ν. ■

Corollary 3.24 *(Extension Theorem for Metric Spaces) Let (X,d) be a metric space and ν be a locally finite continuous valuation on X. Then ν extends uniquely to a regular, τ-smooth Borel measure.*

Remark 3.25 *For metric spaces the property that every finite measure is τ-smooth, is independent of the axioms of set theory. This means that we can assume it or assume its negation without incurring any contradiction [HJ94, 1.34]. Therefore the question whether there exists a probability measure on a metric space that is not the extension of a continuous valuation is undecidable.*

Corollary 3.26 *Let $(X, \mathcal{G})$ be a locally compact sober space and ν be a locally finite continuous valuation on $(X, \mathcal{G}_p)$. Then ν extends uniquely to a regular, τ-smooth Borel measure (on $\mathcal{B}_p X = \sigma(\mathcal{G}_p)$).*

Proof By part (4) of Proposition 3.10 we know that $(X, \mathcal{G}_p)$ is a $T_{3\frac{1}{2}}$ space. ■

Now it is possible to present our second main extension theorem.

Theorem 3.27 ***Extension Theorem for Locally Compact Sober Spaces.*** *Let $(X, \mathcal{G})$ be a locally compact sober space and let ν be a locally finite continuous valuation on X. Then ν can be extended uniquely to a τ-smooth Borel measure.*

In Section 4.3 we show that if ν is finite, then the extension is also a perfect measure.

Proof Once again by Theorem 2.13, it suffices to show the result for finite valuations. Hence we can assume $\nu(X) < \infty$. Let $\overline{\nu}$ denote the unique extension of ν to a content on $\mathcal{A}(\mathcal{G})$. By Corollaries 3.6 and 3.19 it suffices to show that $Lens(X)$ approximates $\overline{\nu}$ on $Cres(X)$. Suppose $G_1 \backslash G_2 \in Cres(X)$ and $\varepsilon > 0$. Without loss of generality we can assume $G_2 \subseteq G_1$. Since X is locally compact, then for each $x \in G_1$ there exist $Q_x \in \mathcal{Q}$ such that $x \in int(Q_x)$ and $Q_x \subseteq G_1$. Therefore

$$G_1 = \bigcup_{x \in G_1} int(Q_x)$$

We have already noted that since $\mathcal{G}$ is a $(\cup f)$-stable paving then the above union can be rewritten as a directed union. Since ν is a continuous valuation, then there exist $x(1),...,x(n) \in G_1$ such that

$$\nu(G_1) - \varepsilon < \nu(\bigcup_{j=1}^{n} int(Q_{x(j)}))$$

Let $Q = \bigcup_{j=1}^{n} Q_{x(j)}$. Since $\mathcal{Q}$ is $(\cup f)$-stable, then $Q \in \mathcal{Q}$, $Q \subseteq G_1$ and $\nu(G_1) - \varepsilon < \nu(int(Q))$. Now notice that G_1 becomes partitioned into four disjoint crescents, namely

$$\begin{aligned}
A &:= int(Q)\backslash G_2 \\
B &:= G_2 \cap int(Q) \\
C &:= G_2 \backslash int(Q) \\
D &:= (G_1 \backslash G_2) \backslash int(Q)
\end{aligned}$$

Let $L := Q\backslash G_2$ (a lens). Then $int(Q)\backslash G_2 \subseteq L \subseteq G_1\backslash G_2$ and

$$\begin{aligned}
\overline{\nu}(G_1\backslash G_2) - \overline{\nu}(int(Q)\backslash G_2) &= \overline{\nu}(A+D) - \overline{\nu}(A) \\
&= \overline{\nu}(D) \\
&\le \overline{\nu}(C+D) \\
&= \overline{\nu}(A+B+C+D) - \overline{\nu}(A+B) \\
&= \overline{\nu}(G_1) - \overline{\nu}(int(Q)) \\
&< \varepsilon
\end{aligned}$$

Since $Lens(X)$ is a monocompact paving, then $\overline{\nu}$ can be extended uniquely to a measure on the σ-algebra generated by $\mathcal{A}(\mathcal{G})$, i.e. the Borel σ-algebra of X. ■

From Proposition 1.10 we obtain.

Corollary 3.28 *(Corollary 2.22) Extension Theorem for Continuous dcpo's*

If D is a continuous dcpo, then every locally finite continuous valuation extends uniquely to a τ-smooth measure on the Borel σ-algebra of $(D, \mathcal{G}_\sigma D)$.

In Section 4.3 we show that if the valuation is finite, then the extension is also a perfect measure. The next result extends Corollary 3.5 in [Law82] which was only given for finite continuous valuations.

Corollary 3.29 *[Nor89, Theorem 7.6](Extension Theorem for Second Countable Locally Compact Sober Spaces) Let $(X, \mathcal{G})$ be a second countable locally compact sober space. Then every locally finite continuous valuation ν extends uniquely to a Radon measure $\overline{\nu}$ on $(X, \mathcal{B}_p X)$.*

Proof By Theorem 3.27 we have that ν extends uniquely to a Borel measure $\overline{\nu}$ on $\mathcal{B}X$. By the Lemma 3.11 $\mathcal{B}X = \mathcal{B}_p X$. By Proposition 3.10-(5) $(X, \mathcal{G}_p)$ is a Polish space. By part (1) of Proposition 1.28 it follows that $\overline{\nu}$ is inner regular. Since ν is locally finite, then $\overline{\nu}$ is Radon. ■

Problem 3.30 *Let $(X, \mathcal{G})$ be a topological space and ν be a (locally) finite continuous valuation on X. Can ν be extended to a Borel measure in any of the following cases:*

1. *X sober but not locally compact.*

2. *X coherent but not locally compact.*

3. *X Hausdorff (recall that Hausdorff spaces are coherent) but not regular.*

4. *Any of the previous plus second or first countability.*

In Section 3.1 of [Hec96], Heckmann defines a valuation ν on $(X, \mathcal{G})$ to be **point continuous** if for all $G \in \mathcal{G}$ and $\varepsilon > 0$ there exists a finite set $I_\varepsilon \subseteq_f G$ such that if $I_\varepsilon \subseteq G_2$, then $\nu(G) - \varepsilon < \nu(G_2)$ for all $G_2 \in \mathcal{G}$. We could weaken this definition and say that a valuation is **$\mathcal{Q}$-continuous** if the same condition holds replacing I_ε with some $Q_\varepsilon \in \mathcal{Q}$. The latter definition is strongly related with $\mathcal{G}$-capacities in the sense of Norberg and Vervaat [NV97] (see next section in this chapter). Point continuous valuations are always continuous [Hec96, Proposition 3.1] and on a locally compact sober space clearly every continuous valuation is $\mathcal{Q}$-continuous. In continuous dcpo's the three notions are equivalent.

Problem 3.31 *For which spaces can point continuous or $\mathcal{Q}$-continuous valuations be extended uniquely to Borel measures? (coherent spaces are probably the best place to start.)*

Problem 3.32 *Characterise spaces where every continuous valuations extends uniquely to a Borel measure.*

3.6 A note on stably locally compact spaces

In this section we compare our extension results to those of Lawson [Law82] and Norberg and Vervaat [NV97] on capacities. The following definitions and results are extracted from [NV97]. Let $(X, \mathcal{G})$ be a topological space and $c : \mathcal{Q} \to \overline{\mathbb{R}}_+$ be a monotone set function. c is a **$\mathcal{Q}$-capacity** if for all $x \in \mathbb{R}_+$ if $c(Q) < x$, then there exists $G \in \mathcal{G}$ such that $Q \subseteq G$ and $c(Q') < x$ for all $Q' \subseteq G$ (outer continuity). If $\mathcal{Q}$ is $(\cap f)$-stable and c satisfies $c(Q_1 \cup Q_2) + c(Q_1 \cap Q_2) = c(Q_1) + c(Q_2)$, then c is called a **modular** $\mathcal{Q}$-capacity. If $(X, \mathcal{G})$ is a locally compact sober space, then a monotone set function c is a $\mathcal{Q}$-capacity if and only if $c(\bigcap_\downarrow Q_i) = \inf_i c(Q_i)$, i.e. it preserves filtered intersections [NV97, Proposition 2.2][Ver97, Theorem 15.4].

Proposition 3.33 *(Tix, Norberg, Vervaat) Let $(X, \mathcal{G})$ be a stably locally compact space. Then,*

1. *if ν is a continuous valuation, then the function $c_\nu : \mathcal{Q} \to \overline{\mathbb{R}}_+$ defined as $c_\nu(Q) := \inf_{G \supseteq Q} \nu(G)$ is a modular $\mathcal{Q}$-capacity satisfying $\nu(G) = \sup_{Q \subseteq G} c_\nu(Q)$,*

2. *if c is a modular $\mathcal{Q}$-capacity, then the function $\nu_c : \mathcal{G} \to \overline{\mathbb{R}}_+$ defined as $\nu_c(G) := \sup_{Q \subseteq G} c(Q)$ is a continuous valuation satisfying $c(Q) = \inf_{G \supseteq Q} \nu_c(G)$.*

If ν and c are as above, then $\nu_{c_\nu} = \nu$ and $c_{\nu_c} = c$.

Proof See[1] Satz 3.4 and Satz 3.5 in [Tix95] and Theorem 5.4 in [NV97]. ■

A topological space $(X, \mathcal{Q})$ has the **upper semicontinuous intersection**[2] property if

(usc) For all $G_1, G_2 \in \mathcal{G}$ and $Q \in \mathcal{Q}$ with $Q \subseteq G_1 \cup G_2$, there exist $Q_1, Q_2 \in \mathcal{Q}$ such that $Q_1 \subseteq G_1$, $Q_2 \subseteq G_2$ and $Q \subseteq Q_1 \cup Q_2$.

The name (usc) is borrowed from Norberg and Vervaat and obeys to the fact that X is (usc) if and only if the mapping $\cap(\_,\_) : \mathcal{F} \times \mathcal{F} \to \mathcal{F}$ given by $\cap(F_1, F_2) = F_1 \cap F_2$ is upper semicontinuous when $\mathcal{F}$ is given the Fell topology. By Proposition 3.14, if X is locally compact, then it has (usc) if $\cap(\_,\_) \in [\mathcal{F}^\lambda \times \mathcal{F}^\lambda \to \mathcal{F}^\sigma]$. Locally compact spaces have (usc) (see part (iv) of Lemma 3.41 below). The following proposition is the main extension result by Norberg and Vervaat.

Proposition 3.34 *(Norberg and Vervaat) Let $(X, \mathcal{G})$ be a sober, (usc) space such that $\mathcal{Q}$ is ($\cap f$)-stable. Then any finite modular $\mathcal{Q}$-capacity c extends uniquely to a compact measure on the σ-algebra generated by $\mathcal{B}X \cup \mathcal{Q}$. The ($\cup f$)-closure of Lens($X$) is a compact paving approximating the measure.*

Proof See Theorem 3.7 in [NV97]. ■

Corollary 3.35 *Let $(X, \mathcal{G})$ be a stably locally compact space. Then every locally finite continuous valuation ν extends uniquely to a compact measure $\overline{\nu}$ on $(X, \sigma(\mathcal{G} \cup \mathcal{G}_{cc}))$. The ($\cup f$)-closure of Lens($X$) is a compact paving approximating this measure.*

Proof If X is stably locally compact, then it is locally compact sober and $\mathcal{Q}$ is ($\cap f$)-stable by definition. We noted that locally compact spaces have (usc). By Proposition 3.33 ν determines a modular $\mathcal{Q}$-capacity $c_\nu(Q) \equiv \inf_{G \supseteq Q} \nu(G)$ satisfying

$$\nu(G) = \sup_{Q \subseteq G} c_\nu(Q).$$

[1] Tix's definition of coherent space corresponds to our definition of stably locally compact space.

[2] Also found in the literature as Reichel's property.

Since ν is locally finite, then by Lemma 2.3 there exists a directed family of open sets $\{G_j\}_{j\in I}$ such that $X = \bigcup^{\uparrow}_{j\in I} G_j$ and $\nu(G_j) < \infty$ for all $j \in I$. It follows that for every $Q \in \mathcal{Q}$ there exists a finite $I_Q \subseteq_f I$ such that $Q \subseteq \bigcup_{j\in I_Q} G_j$. Therefore $c_\nu(Q) \leq \nu(\bigcup_{j\in I_Q} G_j) < \infty$. Hence, the capacity c_ν is finite and by the previous proposition it extends to a compact measure $\overline{c}_\nu$ on the σ-algebra generated by $\mathcal{B}X \cup \mathcal{Q}$. That is, a measure on $\sigma(\mathcal{G} \cup \mathcal{G}_{cc})$ since in stably locally compact spaces $\mathcal{Q}$ is $(\cap)$-stable by part (1) of Lemma 3.12. Fix $G \in \mathcal{G}$, then $c_\nu(Q) = \overline{c}_\nu(Q) \leq \overline{c}_\nu(G)$ for every $Q \in \mathcal{Q}$ with $Q \subseteq G$. Hence,

$$\nu(G) = \sup\nolimits_{Q\subseteq G} c_\nu(Q) \leq \overline{c}_\nu(G)$$

On the other hand, since $\overline{c}_\nu$ is a compact measure approximated by $Lens(X)$ and by Proposition 3.16-(2) we know that $\uparrow L \in \mathcal{Q}$ for all $L \in Lens(X)$, is not difficult to see that

$$\overline{c}_\nu(G) = \sup\nolimits_{Q\subseteq G} c_\nu(Q) = \nu(G).$$

Therefore $\overline{c}_\nu(G) = \nu(G)$ for all $G \in \mathcal{G}$. The conclusion follows by defining $\overline{\nu} := \overline{c}_\nu$. ■

Remark 3.36 *The last corollary is not a consequence of Theorem 3.27, since the extension given by the latter theorem is to a Borel measure on* $\mathcal{B}X = \sigma(\mathcal{G})$ *not on* $\sigma(\mathcal{G}\cup\mathcal{G}_{cc})$. *It is also important to notice that* $\sigma(\mathcal{G}\cup\mathcal{G}_{cc}) \subseteq \sigma(\mathcal{G}_p) = \mathcal{B}_pX$ *but they are not necessarily equal.*

Notice, however, that $\sigma(\mathcal{G} \cup \mathcal{G}_{cc})$ contains the base $B_p = \{G \cap Q^c \mid G \in \mathcal{G}, Q \in \mathcal{Q}\}$ for $\mathcal{G}_p$. This suggests that we could try to extend $\overline{\nu}$ to a patch continuous valuation ν_p. Of course the only possible choice is to let

$$\nu_p(G) := \sup\{\overline{\nu}(\textstyle\bigcup_{i=1}^n b_i) \mid b_i \in B_p, b_i \subseteq G, n \in \mathbb{N}\} \text{ for all } G \in \mathcal{G}_p.$$

This has already been shown by Lawson [Law82] for finite continuous valuations on stably compact spaces.

Proposition 3.37 *(Lawson) Let* $(X, \mathcal{G})$ *be a stably compact space. Then every finite continuous valuation* ν *extends uniquely to a Radon measure* $\overline{\nu}$ *on* $(X, \mathcal{B}_pX)$.

We present an alternative proof that does not use the Riesz representation theorem but where the uniqueness is not guaranteed. First we need the following auxiliary definition and result.

Definition 3.38 *[Top70b, P12] Let* Ω *be any set,* $\mathcal{P}$ *be a* $(\emptyset)$*-paving on* Ω

and $\lambda : \mathcal{P} \to \overline{\mathbb{R}}_+$ be a monotone set function. We say that λ is a ***τ-smooth set function*** *if for every filtered subset $\mathcal{P}^* \subseteq \mathcal{P}$ with $\mathcal{P}^* \downarrow A_0$, we have that*

$$\lambda(A_0) = \inf\{\lambda(A) \mid A \in \mathcal{P}^*\}$$

Notice that a finite Borel measure is τ-smooth if its restriction to the closed sets is a τ-smooth set function.

Lemma 3.39 *(Topsøe) Let $\mathcal{P}_0$ be a $(\emptyset, \cup f, \cap c)$-paving on a set Ω. Suppose that $\lambda : \mathcal{P}_0 \to \mathbb{R}_+$ is a tight*[3] *and τ-smooth set function. Let $\mathcal{P}$ denote the $(\emptyset, \cup f, \cap)$-closure of $\mathcal{P}_0$. Then λ can be extended to a measure μ on $\sigma(\mathcal{P})$ such that*

$$\mu(E) = \sup\{\mu(A) \mid A \subseteq E \text{ and } A \in \mathcal{P}\} \text{ for all } E \in \sigma(\mathcal{P})$$

Proof See Theorem 5.1 in [Top70b]. ■

Proof of Proposition 3.37 (without uniqueness). Let $\mathcal{P}_0$ denote the $(\cup f, \cap c)$-closure of $Lens(X)$ and $\mathcal{P}$ be the $(\emptyset, \cup f, \cap)$-closure of $\mathcal{P}_0$. By Corollary 3.35 we know that ν extends to a compact measure λ on $\sigma(\mathcal{G} \cup \mathcal{G}_{cc})$ and $\mathcal{P}_0$ is a compact paving approximating λ.

1. Claim: $\mathcal{F}_p \subseteq \mathcal{P}$. Let $F \in \mathcal{F}_p$. Since B_p is a base for the patch topology, then

$$F^{\mathsf{c}} = \bigcup\{G \cap Q^{\mathsf{c}} \mid G \cap Q^{\mathsf{c}} \subseteq F^{\mathsf{c}},\, G \in \mathcal{G},\, Q \in \mathcal{Q}\}$$

   Since X is stably compact by assumption, then $X \in \mathcal{Q}$. Hence $G^{\mathsf{c}} = X \cap G^{\mathsf{c}} \in Lens(X)$ for all $G \in \mathcal{G}$. It follows that

$$F = \bigcap\{G^{\mathsf{c}} \cup Q \mid F \subseteq G^{\mathsf{c}} \cup Q\} = \bigcap\{K \mid F \subseteq K,\, K \in \mathcal{P}_0\} \in \mathcal{P}.$$

2. Claim: $\lambda : \mathcal{P}_0 \to \mathbb{R}_+$ is a tight set function. Let K_1 and K_2 be elements of $\mathcal{P}_0$ with $K_2 \subseteq K_1$. Since λ is a measure on $\sigma(\mathcal{G} \cup \mathcal{G}_{cc})$ we have that

$$\begin{aligned} \lambda(K_1) - \lambda(K_2) \quad &= \lambda(K_1 \backslash K_2) \\ &= \sup_{K \subseteq K_1 \backslash K_2}\{\lambda(K) \mid K \in \mathcal{P}_0\} \\ &\text{since } \mathcal{P}_0 \text{ approximates } \lambda \end{aligned}$$

3. Claim: $\lambda : \mathcal{P}_0 \to \mathbb{R}_+$ is a τ-smooth set function. Suppose that $\mathcal{P}^* \subseteq \mathcal{P}_0$ with $\mathcal{P}^* \downarrow A_0$ and $A_0 \in \mathcal{P}_0$. By part (5) of Proposition 3.16 we know that $Lens(X) \subseteq \mathcal{K}_p$, since the patch topology is Hausdorff (Proposition

[3] For this lemma, Topsøe's definition of tightness is $\sup\{\lambda(K) \mid K \subseteq K_1 \backslash K_2\} = \lambda(K_1) - \lambda(K_2)$ whenever $K_2 \subseteq K_1$. This is equivalent to Definition 3.4 since $\mathcal{P}_0$ is a $(\cup f)$-stable paving.

3.10) it follows that $\mathcal{P} \subseteq \mathcal{K}_p$. Let $\varepsilon > 0$ be given. Since $A_0 \in \mathcal{P}_0$ then $A_0^{\mathsf{c}} \in \sigma(\mathcal{G} \cup \mathcal{G}_{cc})$ and there exists $A_1 \in \mathcal{P}_0$ such that $A_1 \subseteq A_0^{\mathsf{c}}$ and $\lambda(A_0^{\mathsf{c}}) - \varepsilon < \lambda(A_1)$. Since λ is finite it follows that

$$\lambda(A_1^{\mathsf{c}}) = \lambda(X) - \lambda(A_1) < \lambda(A_0) + \varepsilon \text{ with } A_0 \subseteq A_1^{\mathsf{c}}.$$

Now notice that A_1^{c} is patch open since A_1 is patch compact. Since Hausdorff spaces are sober and $\mathcal{P} \subseteq \mathcal{K}_p = \mathcal{Q}_p$, then by part (3) of Proposition 3.9 there exists $A_2 \in \mathcal{P}^*$ with $A_0 \subseteq A_2 \subseteq A_1^{\mathsf{c}}$. Hence $\lambda(A_2) < \lambda(A_0) + \varepsilon$. Therefore $\lambda(A_0) = \inf\{\lambda(A) \mid A \in \mathcal{P}^*\}$.

We apply the previous lemma to extend λ to a measure $\overline{\nu}$ on $\sigma(\mathcal{P})$. Since $\mathcal{F}_p \subseteq \mathcal{P} \subseteq \mathcal{K}_p$, then $\mathcal{B}_pX = \sigma(\mathcal{G}_p) = \sigma(\mathcal{P})$ and $\overline{\nu}$ is a Radon measure since $\mathcal{P}$ is an approximating paving for $\overline{\nu}$. ■

Again by Theorem 2.13 we have that if X is stably compact, then every locally finite continuous valuation ν extends uniquely to a Radon measure $\overline{\nu}$ on $(X, \mathcal{B}_pX)$.

Corollary 3.40 ***Extension Theorem for Stably Locally Compact Spaces***

Let $(X, \mathcal{G})$ be a stably locally *compact space. Then every locally finite continuous valuation ν extends uniquely to a Radon measure $\overline{\nu}$ on $(X, \mathcal{B}_pX)$.*

Proof Once again it suffices to show the result when ν is finite. Let $X_\perp$ be the space obtained by adding a bottom element to X. That is, $X_\perp = X \cup \{\perp\}$ with $\perp \sqsubseteq x$ for all $x \in X$. Then $\mathcal{G}X_\perp = \mathcal{G} \cup \{X_\perp\}$ and clearly $(X_\perp, \mathcal{G}X_\perp)$ is a stably compact space. Define $\nu(X_\perp) = \nu(X)$. Apply Proposition 3.37 to extend ν to a Radon measure $\overline{\nu}$ on $\sigma(\mathcal{G}_pX_\perp)$ and notice that $\overline{\nu}(\{\perp\}) = \overline{\nu}(X_\perp) - \overline{\nu}(X) = 0$ since $\overline{\nu}$ is finite. Hence $\overline{\nu}$ restricts to a measure on $\sigma(\mathcal{G}_p) = \mathcal{B}_pX$ that agrees with ν. ■

3.7 Some bitopological properties of the patch space

In this section we are interested in the interplay of four properties that have to do with duality of the patch space as (the symmetrisation of) a bitopological space. The main aspect that we want to highlight, is that stably locally compact spaces are the T_0 spaces that inherit the main properties of Hausdorff spaces. For the rest of this section $(X, \mathcal{G})$ denotes a T_0 topological space. Recall from Section 1.4 that X is locally compact if $\mathcal{K}X$ is a fundamental system of neighborhoods. Since open sets are upper with respect to the specialisation order, this is equivalent to:

- If $x \in G$ and $G \in \mathcal{G}$, then there exist $Q_x \in \mathcal{Q}$ such that $x \in int(Q_x) \subseteq Q_x \subseteq G$.

The following property is implied by local compactness and we call it **weak local compactness**:

(wlc) If $x \notin Q$ and $Q \in \mathcal{Q}$, then there exists $Q_x \in \mathcal{Q}$ such that $Q \subseteq int(Q_x) \subseteq Q_x$ and $x \notin Q_x$.

Equivalently:

($wlc2$) If $x \in Q^{\mathsf{c}}$, then there exist $F_x \in \mathcal{F}$ and $Q_x \in \mathcal{Q}$ such that $x \in Q_x^{\mathsf{c}} \subseteq F_x \subseteq Q^{\mathsf{c}}$.

($wlc3$) If $x \notin Q$ (a patch closed set), then there exist $G_x \in \mathcal{G}$ (a patch open set) and $Q_x \in \mathcal{Q}$ such that $x \in Q_x^{\mathsf{c}}$ (a patch open set) and $Q \subseteq G_x$ with $Q_x^{\mathsf{c}} \cap G_x = \emptyset$.

The last equivalence shows that (wlc) can be thought of as 'half ' of the regularity of the patch topology. The dual 'half' is just the local compactness of $(X, \mathcal{G})$. This is not a coincidence, as we shall see later. We say that X is **weakly Hausdorff** if:

(wH) For all $Q_1, Q_2 \in \mathcal{Q}$ and $G \in \mathcal{G}$ with $Q_1 \cap Q_2 \subseteq G$, there exist $G_1, G_2 \in \mathcal{G}$ such that $Q_1 \subseteq G_1$, $Q_2 \subseteq G_2$ and $G_1 \cap G_2 \subseteq G$.

Recall from the previous section that X has upper semicontinuous intersection if:

(usc) For all $G_1, G_2 \in \mathcal{G}$ and $Q \in \mathcal{Q}$ with $Q \subseteq G_1 \cup G_2$, there exist $Q_1, Q_2 \in \mathcal{Q}$ such that $Q_1 \subseteq G_1$, $Q_2 \subseteq G_2$ and $Q \subseteq Q_1 \cup Q_2$.

The following (rather big) lemma summarises various facts about these properties, most of which are known. We reserve the proof for the end of the section.

Lemma 3.41 *Let $(X, \mathcal{G})$ bet a T_0 topological space ordered with the specialisation order $\sqsubseteq$.*

(i) *General properties*

1. *For all $E \subseteq X$, $sat(E) = \uparrow E$ and $sat_{cc}(E) = \downarrow E$.*
2. *$\mathcal{G}_p$ is T_1.*
3. *If X is T_1, then $\mathcal{K} = \mathcal{Q}$.*

4. *$\mathcal{K}_p \subseteq \mathcal{K}$ and if $K \in \mathcal{K}_p$, then $\uparrow K \in \mathcal{Q}$.*

5. *If X is (wH) and T_1, then X is Hausdorff.*

(ii) *Suppose that X is sober, then $(\mathcal{G}, \subseteq)$ is a continuous lattice if and only if X is locally compact.*

(iii) *Suppose that X is (wlc). Then*

1. *Every compact set has a neighborhood in $\mathcal{Q}$ (in particular every point has a neighborhood in $\mathcal{Q}$).*

2. *Every set in $\mathcal{Q}$ is the intersection of its compact neighborhoods.*

3. *$\mathcal{G}_p$ is Hausdorff and $(X, \mathcal{G}, \mathcal{G}_{cc})$ is pseudo-Hausdorff (see definition below).*

4. *$(X, \mathcal{G}_p, \sqsubseteq)$ is a pospace.*

5. *$(\mathcal{G}_{cc})_{cc} \subseteq \mathcal{G}$ and $\mathcal{Q}_{cc} \subseteq \mathcal{F}$.*

(iv) *Suppose that X is locally compact. Then*

1. *X is (wlc).*

2. *[Tix95, Lemma 3.3-(2)] X has (usc).*

3. *If $K \in \mathcal{K}_p$, then $\downarrow K = cl(K) \in \mathcal{F}$.*

4. *$Lens(X)$ is a fundamental system of neighborhoods for the patch topology.*

(v) *Suppose that X is coherent. Then*

1. *If $\{Q_i\}_{i \in I}$ is any family of sets in $\mathcal{Q}$ and $G \in \mathcal{G}$ with $\bigcap Q_i \subseteq G$, then there exists a finite subset $I_f \subseteq_f I$ such that $\bigcap_{i \in I_f} Q_i \subseteq G$.*

2. *$\mathcal{F} \subseteq \mathcal{Q}_{cc}$ hence $(X, \mathcal{G}_{cc})$ is compact.*

3. *If X is (wlc), then $(X, \mathcal{G}_{cc})$ is locally compact.*

4. *X is (wlc) if and only if X is locally compact and hence stably locally compact.*

(vi) *Suppose that $(X, \mathcal{G})$ is Hausdorff. Then*

1. *X is (wH).*

2. *X is sober.*

3. *X is (wlc) if and only if X is locally compact.*

(*vii*) *Suppose that X is stably locally compact. Then*

1. *[Tix95, Lemma 3.3-(1)] X is (wH).*
2. *[HM81, Theorem 4.8] $(X, \mathcal{G}_p)$ is locally compact.*
3. *$\mathcal{Q}_{cc} = \mathcal{F}$.*
4. *If $(X, \mathcal{G})$ is stably compact (i.e., $X \in \mathcal{Q}$), then $(X, \mathcal{G}_{cc})$ is stably compact.*

Remark 3.42 *Notice that (wlc) implies a number of important properties of the patch space. From part (vii)-4 it follows that if $(X, \mathcal{G})$ is stably compact, then $(X, \mathcal{G}_{cc})$ is sober. Hence by Proposition 3.9-(1) $(X, \sqsubseteq_s^{op})$ is a dcpo. That is, every filtered set in $(X, \sqsubseteq_s)$ has an infimum. Such a property holds for L-domains (see Proposition 4.1.2 in [AJ94]). Nevertheless, it fails to hold for very simple examples of stably locally compact spaces like $(\mathbb{Z} \cup \{\infty\}, \leq)$.*

Now we present some consequences of Lemma 3.41 in the language of bitopological spaces. Our main reference for bitopological spaces is Kopperman [Kop95].

Definition 3.43 *[Kel63] A **bitopological space** $\mathcal{X}$ is a triple $(X, \mathcal{G}, \mathcal{G}^*)$ where X is a set and $\mathcal{G}$, $\mathcal{G}^*$ are topologies in X.*

For the rest of the section $\mathcal{X} = (X, \mathcal{G}, \mathcal{G}^*)$ denotes a bitopological space. The **dual** $\mathcal{X}^*$ of $\mathcal{X}$ is the space $(X, \mathcal{G}^*, \mathcal{G})$. The **symmetrisation** $\mathcal{X}^S$of $\mathcal{X}$ is defined as the space $(X, \mathcal{G}^S)$ where $\mathcal{G}^S = \mathcal{G} \vee \mathcal{G}^*$, that is the topology generated by $\mathcal{G} \cup \mathcal{G}^*$. In this way the patch space $(X, \mathcal{G}_p)$ is the symmetrisation of $(X, \mathcal{G}, \mathcal{G}_{cc})$ and for a poset $(P, \sqsubseteq)$ we have that $(P, \mathcal{G}_\lambda P)$ is the symmetrisation of $(P, \mathcal{G}_\sigma P, \mathcal{G}_l P)$. If $\mathcal{Y} = (Y, \mathcal{H}, \mathcal{H}^*)$ is a bitopological space, then a function $f : \mathcal{X} \to \mathcal{Y}$ is continuous if and only if both $f : (X, \mathcal{G}) \to (Y, \mathcal{H})$ and $f : (X, \mathcal{G}^*) \to (Y, \mathcal{H}^*)$ are continuous. When we refer to the sets and properties of $(X, \mathcal{G}, \mathcal{G}^*)$, the absence or presence of the superscript '$*$' indicates to which topology we are referring to. For example, if $A \subseteq X$, then $int(A)$ is the interior of A in $(X, \mathcal{G})$, while $cl^*(A)$ is the closure of A in $(X, \mathcal{G}^*)$; if we say that F is a *-closed set, it means that $F \in \mathcal{F}^*$. We define the bitopological space $\mathfrak{I} := ([0,1], \{(a, \infty) \cap [0,1] \mid a \in \mathbb{R}\}, \{(-\infty, b) \cap [0,1] \mid b \in \mathbb{R}\})$. If we consider the poset $([0,1], \leq)$, then $\mathfrak{I} = ([0,1], \mathcal{G}_\sigma[0,1], \mathcal{G}_l[0,1])$. Obviously the symmetrisation $\mathfrak{I}^S$ is the unit interval with its usual topology. A bitopological space $\mathcal{X} = (X, \mathcal{G}, \mathcal{G}^*)$ is:

- **Normal** if for all F_1^* *-closed and G_1 open with $F_1^* \subseteq G_1$, there exist F_2^* *-closed and G_2 open such that $F_1^* \subseteq G_2 \subseteq F_2^* \subseteq G_1$.

- **Completely regular** if for all $x \in G$ with G open, there exists a continuous function $f : \mathcal{X} \to \mathfrak{I}$ such that $f(x) = 1$ and $f(y) = 0$ for all $y \in G^{\mathsf{c}}$.
- **Regular** if for all $x \in G$ with G open, there exist an open G_x and a *-closed set F_x^* such that $x \in G_x \subseteq F_x^* \subseteq G$.
- **Pseudo-Hausdorff** if for $y \notin cl(x)$, there exist an open G and a *-open G^* such that $y \in G$ and $x \in G^*$ with $G \cap G^* = \emptyset$.
- **Weakly symmetric** if $x \notin cl(y)$ implies that $y \notin cl^*(x)$.
- T_0 if $\mathcal{G}^S$ is T_0.
- $T_1 = T_0+$ weakly symmetric, Hausdorff$= T_2 = T_0+$ pseudo-Hausdorff, $T_3 = T_0+$ regular, $T_{3\frac{1}{2}} = T_0+$ completely regular, $T_4 = T_1+$ normal.

Although these definitions might seem a bit arbitrary at first sight, the idea is that $(X, \mathcal{G}, \mathcal{G})$ satisfies any of the properties above (as a bitopological space) if and only if $(X, \mathcal{G})$ satisfies the corresponding property as a topological space. For a thorough exposition of the relation among these properties the reader is referred to [Kop95]. The **de Groot dual** [dG67] of a topology $\mathcal{G}$ on X is defined as the cocompact topology $\mathcal{G}_{cc}$ on X. For the class of pseudo-Hausdorff spaces the de Groot dual is indeed a topological dual since $((\mathcal{G}_{cc})_{cc})_{cc} = \mathcal{G}_{cc}$ (we also have that $(\mathcal{G}_{cc})_{cc} \subseteq \mathcal{G}$). Now we are in possibility of formulating some consequences of Lemma 3.41. The next theorem shows precisely in which sense are (wlc) and local compactness dual properties.

Theorem 3.44 *Let $(X, \mathcal{G})$ be a topological space. Then*

1. *$(X, \mathcal{G})$ is (wlc) if and only if $(X, \mathcal{G}_{cc}, \mathcal{G})$ is regular.*
2. *If $(X, \mathcal{G})$ is locally compact, then $(X, \mathcal{G}, \mathcal{G}_{cc})$ is regular. The converse holds if $(X, \mathcal{G})$ is coherent.*
3. *If $(X, \mathcal{G})$ is coherent, then*
$$(X, \mathcal{G}, \mathcal{G}_{cc}) \text{ is regular if and only if } (X, \mathcal{G}_{cc}, \mathcal{G}) \text{ is regular.}$$
4. *If $(X, \mathcal{G})$ is stably locally compact, then $(X, \mathcal{G}, \mathcal{G}_{cc})$ is normal.*

Proof Recall that $\mathcal{Q}$ is base for the closed sets of $(X, \mathcal{G}_{cc})$.

1. This follows from $(wlc2)$ and the definitions since each member of $\mathcal{G}_{cc}$ is of the form $\bigcup_{i \in I} Q_i^{\mathsf{c}}$ with $Q_i \in \mathcal{Q}$ for all $i \in I$ (an arbitrary set of indices).

2. The first claim is obvious since $\mathcal{Q} \subseteq \mathcal{F}_{cc}$. For the second claim notice that by Lemma 3.12-(1) we have that $\mathcal{F}_{cc} \backslash \{X\} \subseteq \mathcal{Q}$. Hence we just need to check the case where $x \in G = X$. If $X = \uparrow x$, then $X \in \mathcal{Q}$ and we are finished. Otherwise, there exists $y \in X$ with $y \notin \uparrow x$; that is $x \notin cl(y)$. Hence then there exists $G_2 \in \mathcal{G}$ such that $x \in G_2$ and $y \notin G_2$. Then, by regularity of $(X, \mathcal{G}, \mathcal{G}_{cc})$, there exists $F \in \mathcal{F}_{cc}$ such that $x \in int(F) \subseteq F \subseteq G_2$. Since $y \notin F$, then $F \neq X$. Hence $F \in \mathcal{Q}$ and the claim follows.

3. This follows from parts (1), (2) and part (v)-4 of Lemma 3.41.

4. We noted in the proof of part (2) that $\mathcal{F}_{cc} \backslash \{X\} \subseteq \mathcal{Q}$. Therefore, the claim follows by part (1) of Proposition 3.10. ■

At this point is not difficult to guess that (wH) and (usc) should be particular instances on $(X, \mathcal{G}, \mathcal{G}_{cc})$ of two general properties for bitopological spaces.

In order to motivate their definition consider the following definition from lattice theory. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be sublattices of a Boolean algebra $(\mathcal{B}, 0, 1, \vee, \wedge, ')$.

- We say that $\mathcal{L}_1$ **separates** $\mathcal{L}_2$ if for all $a_2, b_2 \in \mathcal{L}_2$ such that $a_2 \wedge b_2 = 0$, there exist $a_1, b_1 \in \mathcal{L}_1$ such that $a_2 \leq a_1$, $b_2 \leq b_1$ and $a_1 \wedge b_1 = 0$.
- We say that $\mathcal{L}_1$ **coseparates** $\mathcal{L}_2$ if for all $a_2, b_2 \in \mathcal{L}_2$ such that $a_2 \wedge b_2 = 0$, there exist $a_1, b_1 \in \mathcal{L}_1$ such that $a_2 \leq (a_1)'$, $b_2 \leq (b_1)'$ and $(a_1)' \wedge (b_1)' = 0$.

We define a bitopological space $\mathcal{X} = (X, \mathcal{G}, \mathcal{G}^*)$ to be:

- **Separating** if for all F_1^*, F_2^* *-closed and G open with $F_1^* \cap F_2^* \subseteq G$, there exist G_1 and G_2 open such that $F_1^* \subseteq G_1$, $F_2^* \subseteq G_2$ and $G_1 \cap G_2 \subseteq G$.
- **Coseparating** if for all G_1, G_2 open and F^* *-closed with $F^* \subseteq G_1 \cup G_2$, there exist F_1^*, F_2^* *-closed such that $F_1^* \subseteq G_1$, $F_2^* \subseteq G_2$ and $F^* \subseteq F_1^* \cup F_2^*$.

Theorem 3.45 *Let $\mathcal{X} = (X, \mathcal{G}, \mathcal{G}^*)$ be a bitopological space. Consider $\mathcal{G}$, $\mathcal{F}$, $\mathcal{F}^*$ as sublattices*[4] *of $(2^X, \emptyset, X, \cup, \cap, ^c)$. Then,*

1. *$(X, \mathcal{G}, \mathcal{G}^*)$ is separating if and only if $(X, \mathcal{G}^*, \mathcal{G})$ is coseparating.*

[4]For this theorem, $\mathcal{F}$ and $\mathcal{F}^*$ are ordered with $\subseteq$. In the rest of the thesis the assumed order for closed sets is $\supseteq$.

2. If $\mathcal{X}$ is separating, then $\mathcal{G}$ separates $\mathcal{F}^*$.

3. If $\mathcal{X}$ is coseparating, then $\mathcal{F}^*$ coseparates $\mathcal{F}$.

4. $(X, \mathcal{G}, \mathcal{G})$ is separating (or coseparating) if and only if $(X, \mathcal{G})$ is normal.

Proof

1. Is immediate from de Morgan laws.

2. Follows by taking $G = \emptyset$.

3. Follows by taking $F^* = X$.

4. $\boxed{\Rightarrow}$ Is immediate from the definition of separating with $G = \emptyset$. $\boxed{\Leftarrow}$ (Same argument as in the proof of part (vi)-1 of Lemma 3.41 below) Suppose that $F_1, F_2 \in \mathcal{F}$ and $G \in \mathcal{G}$ are such that $F_1 \cap F_2 \subseteq G$. Then $F_1 \cap G^{\mathsf{c}}$ and $F_2 \cap G^{\mathsf{c}}$ are closed and $(F_1 \cap G^{\mathsf{c}}) \cap (F_2 \cap G^{\mathsf{c}}) = \emptyset$. Since X is normal, then there exist open sets $G_1', G_2' \in \mathcal{G}$ such that $F_1 \cap G^{\mathsf{c}} \subseteq G_1'$ and $F_2 \cap G^{\mathsf{c}} \subseteq G_2'$ with $G_1' \cap G_2' = \emptyset$. Let $G_1 = G_1' \cup G$ and $G_2 = G_2' \cup G$. Then $F_1 \subseteq G_1$, $F_2 \subseteq G_2$ and $G_1 \cap G_2 = G$. ■

In spite of part (4) in the previous theorem, it is interesting to note that the separation (or coseparation) of $\mathcal{X}$ does not imply in general the normality of the symmetrisation $\mathcal{X}^S$ as the following example shows.

Example 3.46 *(Kopperman) Let $(Y, \mathcal{G})$ be any Hausdorff space which is not normal. Let $\mathcal{G}^*$ be the cofinite topology on Y. That is, $\mathcal{F}^* = 2^Y_f \cup \{\emptyset, Y\}$. Since in a Hausdorff space all finite subsets are closed, then $\mathcal{F}^* \subseteq \mathcal{F}$. Hence the symmetrisation of $(Y, \mathcal{G}, \mathcal{G}^*)$ is $(Y, \mathcal{G})$. We claim that $(Y, \mathcal{G}, \mathcal{G}^*)$ is separating. Suppose that F_1^*, F_2^* are *-closed and G is open with $F_1^* \cap F_2^* \subseteq G$. If $F_1^* = Y$, then let $G_1 := Y$ and $G_2 := G$. If both F_1^* and F_2^* are finite, then for all $x \in F_1^* \cup F_2^*$ find $H_x \in \mathcal{G}$ such that $x \in H_x$ and $H_x \cap H_y = \emptyset$ if $x \neq y$. Now define for all $x \in F_1^* \cup F_2^*$*

$$G_x := \begin{cases} H_x \cap G & \text{if } x \in G \\ H_x & \text{otherwise} \end{cases}$$

Let $G_i := \bigcup_{x \in F_i^*} G_x$ for $i = 1, 2$. Clearly $F_i^* \subseteq G_i$ and it is not difficult to see that $G_1 \cap G_2 \subseteq G$. Hence $(Y, \mathcal{G}, \mathcal{G}^*)$ is separating.

In the previous example, if there exists $x \in Y$ such that all the neighborhoods of x have an infinite number of points, then by choosing $F_1^* := \{x\}$ and $G_1 \neq Y$ we can see that $(Y, \mathcal{G}, \mathcal{G}^*)$ is separating but not normal. The irrational slope topology (example 75 in [SS78]) is an example of such a space.

The next example shows that neither the normality of $\mathcal{X}$ implies that $\mathcal{X}$ is separating.

Example 3.47 *Let* $\mathcal{G} := \{\emptyset, \mathbb{R}, \{0\}, \mathbb{R}\backslash\{0\}\}$ *and* $\mathcal{G}^* := \mathcal{T}(\{(a,0) \mid a < 0\} \cup \{(0,b) \mid 0 < b\})$. *We claim that* $(\mathbb{R}, \mathcal{G}, \mathcal{G}^*)$ *is normal but not separating. Notice that the sets in* $\mathcal{F}^*$ *fall in one of the following categories.*

- $\emptyset, \mathbb{R}$ *and* $\{0\}$.
- $\mathbb{R}\backslash(a,0)$ *or* $(-\infty, a] \cup \{0\}$ *with* $a < 0$.
- $\mathbb{R}\backslash(0,b)$ *or* $\{0\} \cup [b, \infty)$ *with* $0 < b$.
- $(-\infty, a] \cup \{0\} \cup [b, \infty)$ *with* $a < 0 < b$.

Hence, 0 *is an element of all the nonempty* **-closed sets. Let* $E_1^* := (-\infty, -1] \cup \{0\}$, $E_2^* := \{0\} \cup [1, \infty)$ *and* $G := \{0\}$. *Clearly,* $E_1^* \cap E_2^* \subseteq G$ *but the only open set (in* $\mathcal{G}$*) that contains* E_i^* *is* $\mathbb{R}$ *for* $i = 1, 2$. *Therefore* $(\mathbb{R}, \mathcal{G}, \mathcal{G}^*)$ *is not separating. On the other hand suppose that* F_1^* *is* **-closed and* G_1 *is open with* $F_1^* \subseteq G_1$. *Consider the following cases.*

- *If* $F_1^* = \emptyset$ *or* $G_1 = \emptyset$, *then let* $G_2 := F_2^* := \emptyset$.
- *If* $F_1^* = \{0\}$, *then let* $G_2 := F_2^* := \{0\}$.
- *Otherwise* F_1^* *is an unbounded subset of* $\mathbb{R}$, *hence* $G_1 = \mathbb{R}$. *Let* $G_2 := F_2^* := \mathbb{R}$.

We conclude that $(\mathbb{R}, \mathcal{G}, \mathcal{G}^*)$ *is normal.*

In the last example, notice that the topology $\mathcal{G}^*$ is T_0 but the topology $\mathcal{G}$ is not. We show that $\mathcal{G}$ can be refined to a T_0 topology $\mathcal{G}'$. This has to be done carefully. If we only add intervals of the form $(-a, a)$ with $a > 0$, then $a \in cl(-a)$ and $-a \in cl(a)$ but $a \neq -a$. If we add all the intervals of the form (a, b) with $0 \in (a, b)$, then $(-1, \infty)$ is an open set, different from $\mathbb{R}$, which contains $E_2^* = \{0\} \cup [1, \infty)$; hence the case analysis becomes complicated. Therefore, let $\mathcal{G}' := \mathcal{T}(\mathcal{G} \cup \{(-\frac{a}{2}, a) \mid a \in \mathbb{R}\} \cup \{(-a, \frac{a}{2}) \mid a \in \mathbb{R}\})$, then $\mathcal{G}'$ is T_0 and the same argument as above shows that $(\mathbb{R}, \mathcal{G}', \mathcal{G}^*)$ is still normal but not separating.

The separation properties have the following implications in the case that the second topology is the de Groot dual of the first.

Theorem 3.48 *Let* $(X, \mathcal{G})$ *be a topological space. Then,*

1. *If* $(X, \mathcal{G}, \mathcal{G}_{cc})$ *is separating, then* $(X, \mathcal{G})$ *is* (wH).

2. *Suppose that $(X, \mathcal{G})$ is coherent. If $(X, \mathcal{G}, \mathcal{G}_{cc})$ is coseparating, then $(X, \mathcal{G})$ has (usc).*

3. *If $(X, \mathcal{G})$ is coherent and compact. Then,*

   - *$(X, \mathcal{G}, \mathcal{G}_{cc})$ is separating if and only if $(X, \mathcal{G})$ is (wH).*
   - *$(X, \mathcal{G}, \mathcal{G}_{cc})$ is coseparating if and only if $(X, \mathcal{G})$ is (usc).*

4. *If $(X, \mathcal{G})$ is stably compact, then $(X, \mathcal{G}, \mathcal{G}_{cc})$ and $(X, \mathcal{G}_{cc}, \mathcal{G})$ are both separating and coseparating.*

Proof Recall that $\mathcal{Q}$ is a base for the closed sets of $(X, \mathcal{G}_{cc})$. Part (1) is obvious since $\mathcal{Q} \subseteq \mathcal{F}_{cc}$. For parts (2) and (3) notice that by Lemma 3.12-(1) we have that $\mathcal{F}_{cc} \backslash \{X\} \subseteq \mathcal{Q}$. If X is compact, then obviously $\mathcal{F}_{cc} = \mathcal{Q}$. Part (4) follows from part (3) and parts (iv)-2 and (vii)-1 of Lemma 3.41. ∎

The reader is referred to Section 4 in [Kop95] for more results on bitopological properties of stably compact spaces[5]. See also [KKW98].

Problem 3.49 *Suppose that $(X, \mathcal{G})$ is a topological space and ν is a continuous valuation on X. What conditions on $\mathcal{X} = (X, \mathcal{G}, \mathcal{G}^*)$ ensure that ν extends uniquely to a Borel measure in $\sigma(\mathcal{G} \vee \mathcal{G}^*)$? What happens when $\mathcal{X}$ is separating, normal or completely regular?*

Proof of Lemma 3.41.

(i) General properties.

1. $\boxed{\subseteq}$ Suppose that $x \notin \uparrow E$. That is, for each $y \in E$ we have that $y \notin cl(x)$. Therefore, there exists $G_y \in \mathcal{N}(y)$ such that $x \notin G_y$. Then $G := \bigcup_{y \in E} G_y$ is an open set such that $E \subseteq G$ and $x \notin G$. It follows that $x \notin sat(E)$. $\boxed{\supseteq}$ If $x \in \uparrow E$, then there exists $y \in E$ such that $y \sqsubseteq x$. That is, $y \in cl(x)$. Therefore any nbd of y contains x, hence $x \in sat(y) \subseteq sat(E)$.

   For the second part notice that $cl_{cc}(x) = \uparrow x$, hence the specialisation order in $(X, \mathcal{G}_{cc})$ is precisely $\sqsubseteq_s^{op}$. Therefore $sat_{cc}(E) = \uparrow_{cc} E = \downarrow E$.

2. Let x, y be different elements of X. Since X is T_0, then without loss of generality, there exists $G \in \mathcal{G}$ such that $y \in G$ and $x \notin G$. Therefore $x \notin \uparrow y$ since G is upper. Then $(\uparrow y)^{\mathsf{c}}$ and G are patch open sets that separate x and y.

[5] Strictly speaking for skew compact spaces, which is shown to be an equivalent notion.

3. If X is T_1, then $cl(x) = \{x\}$ for all $x \in X$. Hence $x \sqsubseteq y$ if and only if $x = y$. That is, for all $E \subseteq X$ we have that $E = \uparrow E$ and by part (1) it follows that all the sets are saturated.

4. $\mathcal{K}_p \subseteq \mathcal{K}$ is obvious since the patch topology refines $\mathcal{G}$, hence every $\mathcal{G}$-covering for a set is also an $\mathcal{G}_p$-covering. Suppose that $K \in \mathcal{K}_p$ and let $\mathcal{C}$ be an open $\mathcal{G}$-covering for $\uparrow K$. Since $K \subseteq \uparrow K$, then $\mathcal{C}$ is also a open $\mathcal{G}_p$-covering for K, and hence it admits a finite subcovering $\mathcal{C}_f$. Since the sets in $\mathcal{G}$ are upper, then $\mathcal{C}_f$ is also a covering for $\uparrow K$.

5. Let $x, y \in X$ with $x \neq y$. We observed that for T_1 spaces $\mathcal{K} = \mathcal{Q}$, then $\{x\}, \{y\} \in \mathcal{Q}$ and $\{x\} \cap \{y\} \subseteq \emptyset$. Therefore, there exist $G_x, G_y \in \mathcal{G}$ such that $\{x\} \subseteq G_x$, $\{y\} \subseteq G_y$ and $G_x \cap G_y \subseteq \emptyset$.

(ii) See Theorem V-5.6 in [GHK+80].

(iii) $(X, \mathcal{G})$ is (wlc).

1. Let K be any set in $\mathcal{K}$. By part (i)-4 we know that $\uparrow K \in \mathcal{Q}$. If $\uparrow K = X$, then we are finished. Otherwise there exists $x \notin \uparrow K$. Then by hypothesis there exists $Q_x \in \mathcal{Q}$ such that $\uparrow K \subseteq int(Q_x) \subseteq Q_x$.

2. Let $Q \in \mathcal{Q}$. Suppose that $x \notin Q$, then by assumption there exists $Q_x \in \mathcal{Q}$ such that $Q \subseteq int(Q_x)$ and $x \notin Q_x$ hence $Q = \bigcap\{Q_x \mid x \in Q^{\mathsf{c}}\}$.

3. (cf. [HM81, Corollary 4.6]) Let x, y be different elements of X. Since X is T_0, without loss of generality, we can assume $y \notin cl(x)$. Then $x \notin \uparrow y = Q$. By $(wlc3)$ there exist $G_x \in \mathcal{G}$ with $\uparrow y \subseteq G_x$ and $Q_x \in \mathcal{Q}$ with $x \in Q_x^{\mathsf{c}} \in \mathcal{G}_{cc}$ such that $G_x \cap Q_x^{\mathsf{c}} = \emptyset$. Therefore $\mathcal{G}_p$ is Hausdorff. This argument also shows that $(X, \mathcal{G}, \mathcal{G}_{cc})$ is pseudo-Hausdorff.

4. From the proof of the previous statement it follows that $(y, x) \in (int(Q_x) \times Q_x^{\mathsf{c}}) \in \mathcal{G}_p \otimes \mathcal{G}_p$. Furthermore, since $int(Q_x)$ is upper and Q_x^{c} is lower, then $int(Q_x) \times Q_x^{\mathsf{c}}$ is a subset of $(R_{\sqsubseteq})^{\mathsf{c}}$. Therefore $(R_{\sqsubseteq})^{\mathsf{c}}$ is open.

5. By part (iii)-3 we know that $(X, \mathcal{G}, \mathcal{G}_{cc})$ is pseudo-Hausdorff. By Lemma 4.6-(b) in [Kop95] it follows that $(\mathcal{G}_{cc})_{cc} \subseteq \mathcal{G}$. Suppose that $Q \in \mathcal{Q}_{cc}$. Then $Q^{\mathsf{c}} \in (\mathcal{G}_{cc})_{cc} \subseteq \mathcal{G}$. We conclude that $Q \in \mathcal{F}$.

(iv) $(X, \mathcal{G})$ is locally compact.

1. Suppose that $x \in X$ and $Q \in \mathcal{Q}$ with $x \notin Q$. Since Q is saturated and X is T_0, then for every $y \in Q$ there exists $G_y \in \mathcal{G}$ such that $y \in G_y$ and $x \notin G_y$. By hypothesis, there exist $Q_y \in \mathcal{Q}$ such that $y \in int(Q_y) \subseteq Q_y \subseteq G_y$. Then $\{int(Q_y)\}_{y \in Q}$ is an open covering for Q. Since Q is compact we can extract a finite subcovering $\{int(Q_{y_k})\}_{k=1}^{n}$. Let $G_x := \bigcup_{k=1}^{n} int(Q_{y_k})$, and $Q_x := \bigcup_{k=1}^{n} Q_{y_k}$. Note that $Q_x \in \mathcal{Q}$ since $\mathcal{Q}$ is closed under finite unions. Therefore $Q \subseteq G_x \subseteq Q_x$ and $x \notin Q_x$.

2. Suppose that $G_1, G_2 \in \mathcal{G}$ and $Q \in \mathcal{Q}$ are such that $Q \subseteq G_1 \cup G_2$. Since X is locally compact, then for all $x \in Q \cap G_1$ there exist $Q_x \in \mathcal{Q}$ such that $x \in int(Q_x) \subseteq Q_x \subseteq G_1$. A similar thing happens for all $y \in Q \cap G_2$. Therefore the paving

$$\mathcal{C} = \{int(Q_x) \cup int(Q_y) \mid x \in Q \cap G_1 \text{ and } y \in Q \cap G_2\}$$

is an open covering for Q satisfying $\bigcup \mathcal{C} \subseteq G_1 \cup G_2$. Since Q is compact then we can extract a finite subcovering

$$\mathcal{C}_0 = \{int(Q_{x_i}) \cup int(Q_{y_i}) \mid \begin{array}{l} x_i \in Q \cap G_1 \text{ and} \\ y_i \in Q \cap G_2 \text{ for } i = 1, \ldots, n\}. \end{array}$$

Let $Q_1 := \bigcup_{i=1}^{n} Q_{x_i}$ and $Q_2 := \bigcup_{i=1}^{n} Q_{y_i}$, then $Q_1 \subseteq G_1$, $Q_2 \subseteq G_2$ and $Q \subseteq \bigcup \mathcal{C}_0 \subseteq Q_1 \cup Q_2$.

3. (cf. [AJ94, Lemma 6.2.20]) Suppose that $K \in \mathcal{K}_p$. Since closed sets are lower and $\downarrow K$ is the minimum lower set containing K it suffices to show that $\downarrow K$ is closed. Suppose that $x \notin \downarrow K$, then, for each $y \in K$ we have that $x \notin cl(y)$. Therefore, there exists $G_y \in \mathcal{G}$ such that $x \in G_y$ and $y \notin G_y$. Since X is locally compact, then there exist $Q_y \in \mathcal{Q}$ with $x \in int(Q_y) \subseteq Q_y \subseteq G_y$ and $y \notin Q_y$. It follows that $\{Q_y^{\mathsf{c}}\}_{y \in K}$ is a open $\mathcal{G}_p$-covering for K. Therefore we can extract a finite subcovering $\{Q_{y_k}^{\mathsf{c}}\}_{k=1}^{n}$. Then $\downarrow K \subseteq \bigcup_{k=1}^{n} Q_{y_k}^{\mathsf{c}}$ since the complements of sets in $\mathcal{Q}$ are lower. Then $G = \bigcap_{k=1}^{n} int(Q_{y_k}) \subseteq \bigcap_{k=1}^{n} Q_{y_k} \subseteq (\downarrow K)^{\mathsf{c}}$. Note that $x \in G \in \mathcal{G}$, hence $(\downarrow K)^{\mathsf{c}}$ is open.

4. See part (1) of Lemma 5.23 below (sobriety is not required).

(v) $(X, \mathcal{G})$ is sober and $\mathcal{Q}$ is $(\cap f)$-stable.

1. Rewrite $\bigcap_{i \in I} Q_i$ as $\bigcap_{\downarrow} \{\bigcap_{i \in I_f} Q_i \mid I_f \subseteq_f I\}$, then apply part (3) of Proposition 3.9.

2. Let $F \in \mathcal{F}$. For proving that $F \in \mathcal{K}_{cc}$, it suffices to show that

every covering $\{Q_i^c\}_{i\in I}$ of cocompact basic opens for F has a finite subcovering. But $\bigcap_{i\in I} Q_i \subseteq F^c$, then by part (v)-1 we can find a finite subset $I_f \subseteq_f I$ such that $\bigcap_{i\in I_f} Q_i \subseteq F^c$. Hence $\{Q_i^c\}_{i\in I_f}$ is a finite open $\mathcal{G}_{cc}$-subcovering for F. By part (i)-1 we have that $sat_{cc}(F) = \downarrow F = F$ and the claim follows.

3. It suffices to show the result for basic open sets of $\mathcal{G}_{cc}$. Let $Q \in \mathcal{Q}$ and suppose that $x \in Q^c$. Then by $(wlc2)$ there exist $F_x \in \mathcal{F}$ and $Q_x \in \mathcal{Q}$ such that $x \in Q_x^c \subseteq F_x \subseteq Q^c$. Then $Q_x^c \in \mathcal{G}_{cc}$ and by part (v)-2 we know that $F_x \in \mathcal{Q}_{cc}$.

4. Suppose that $y \in G \in \mathcal{G}$. Since $\uparrow y \in \mathcal{Q}$, then by (wlc) for all $x \notin \uparrow y$ there exists $Q_x \in \mathcal{Q}$ such that $\uparrow y \subseteq int(Q_x)$ and $x \notin Q_x$. Hence $\uparrow y = \bigcap\{Q_x \mid x \notin \uparrow y\} \subseteq G$. Then by part (v)-1 there exists $n \in \mathbb{N}$ such that $Q := \bigcap_{j=1}^{n} Q_{x_j} \subseteq G$. It follows that $y \in int(Q)$ and $Q \subseteq G$ with $Q \in \mathcal{Q}$.

(vi) $(X, \mathcal{G})$ is Hausdorff.

1. Suppose that $Q_1, Q_2 \in \mathcal{Q}$ and $G \in \mathcal{G}$ are such that $Q_1 \cap Q_2 \subseteq G$. Since Q_1, Q_2 are compact, then $Q_1 \backslash G$ and $Q_2 \backslash G$ are compact. Notice that $(Q_1 \backslash G) \cap (Q_2 \backslash G) = (Q_1 \cap Q_2) \cap G^c = \emptyset$. Since X is Hausdorff, then [Dug66, XI-1.5-(b)] there exist open sets $G_1', G_2' \in \mathcal{G}$ such that $Q_1 \backslash G \subseteq G_1'$ and $Q_2 \backslash G \subseteq G_2'$ with $G_1' \cap G_2' = \emptyset$. Let $G_1 = G_1' \cup G$ and $G_2 = G_2' \cup G$. Then $Q_1 \subseteq G_1$, $Q_2 \subseteq G_2$ and $G_1 \cap G_2 = G$.

2. Suppose that $F \in \mathcal{F}$ and $x, y \in F$ with $x \neq y$. Then by hypothesis there exist open sets G_x, G_y such that $x \in G_x, y \in G_y$ and $G_x \cap G_y = \emptyset$. Then $F \subseteq G_x^c \cup G_y^c = X$. Since $F \subsetneq G_x^c$ and $F \subsetneq G_y^c$ we have that F is not irreducible. Hence the only irreducible sets are singletons.

3. The 'if' part is immediate from part (iv)-1. For the 'only if' part, (iii)-1 implies that every point has a compact neighborhood[6]. For Hausdorff spaces this property is equivalent to local compactness [Dug66, Theorem XI-(6.2)].

(vii) $(X, \mathcal{G})$ is sober, locally compact and $\mathcal{Q}$ is $(\cap f)$-stable.

1. Suppose that $Q_1, Q_2 \in \mathcal{Q}$ and $G \in \mathcal{G}$ are such that $Q_1 \cap Q_2 \subseteq G$. Let $\mathcal{C}_i = \{C \in \mathcal{Q} \mid Q_i \subseteq int(C)\}$ for $i = 1, 2$. Then by parts

[6]This is the usual definition of local compactness in Hausdorff spaces.

(iv)-1 and (iii)-2 we have that $Q_i = \bigcap \mathcal{C}_i$. Furthermore, since $\mathcal{Q}$ is $(\cap f)$-stable we have that $\mathcal{C}_i {\downarrow} Q_i$. Hence $\mathcal{C} := \{C_1 \cap C_2 \mid C_i \in \mathcal{C}_i\}$ is such that $\mathcal{C}{\downarrow}(Q_1 \cap Q_2)$. Since $Q_1 \cap Q_2 \subseteq G$, then by part (3) of Proposition 3.9 there exist $C_1 \cap C_2 \in \mathcal{C}$ such that $C_1 \cap C_2 \subseteq G$. Let $G_i = int(C_i)$ for $i = 1, 2$.

2. By part (iv)-4 we have that $Lens(X)$ is a fsn for the patch topology. By part (5) of Proposition 3.16 we have that $Lens(X) \subseteq \mathcal{K}_p$ and the conclusion follows.

3. This is immediate from parts (iii)-5 and (v)-2.

4. We use the terminology in [Kop95]. From our Proposition 1.12 and Theorem 4.11 in [Kop95] we have that $(X, \mathcal{G})$ is stably compact $\Leftrightarrow$ $(X, \mathcal{G})$ is skew compact. By Theorem 3.7 in [Kop95] if $(X, \mathcal{G})$ is skew compact, then $(X, \mathcal{G}_{cc})$ is skew compact and the conclusion follows. ■

4. Perfectness, disintegration and factorization

4.1 Summary

Suppose that $(X, \mathcal{G})$ is a locally compact and sober topological space. We show that the extension of a finite continuous valuation on X to a Borel measure given by Theorem 3.27, is a perfect measure (Corollary 4.15). In particular, every finite continuous valuation on a continuous dcpo extends to a perfect measure. Topsøe has shown that if a content μ on an algebra $\mathcal{A}$ can be approximated by a monocompact paving (see below), then the extension of μ to a measure on $\sigma(\mathcal{A})$ is perfect. We extend this result to semialgebras (Theorem 4.13) using similar techniques via projective limits of perfect measure spaces. From Section 3.6 we know that on stably locally compact spaces the extension of a finite valuation to a measure is compact. The compactness and perfectness of these extensions allow us to establish the existence of disintegrations (regular conditional probability distributions) and factorizations for certain classes of spaces (Theorem 4.27).

Suppose that $\langle \mu_j \rangle_{j \in J}$ is a directed set of simple valuations on X and $\mu = \sup_{j \in J} \mu_j$ is finite. We know from Chapters 2 and 3 that μ extends uniquely to a measure. Notice that the extensions of the μ_j are compact measures approximated by their finite set of support points $|\mu_j|$. The finite sets $|\mu_j|$ (although compact) might not be Borel sets of $\mathcal{G}$, nevertheless they are closed and compact in $(X, \mathcal{G}_p)$ since the patch space is Hausdorff. Hence the extension of μ_j to a measure on $\mathcal{B}_p X$ gives a Radon measure for all $j \in J$. The question whether the extension of μ would also result in a compact measure, puzzled the author for long time. A positive answer would imply that every finite continuous valuation on a continuous dcpo extends to a compact measure. On the other hand perfect measures, as we shall see in the next section, are 'almost compact' measures. We do not have an example of a continuous valuation on a locally compact sober space whose extension is perfect but not compact.

The motivation for studying the existence of regular conditional probabilities on continuous dcpo's arose from the work of Edalat [Eda99]. In [Eda99] it was shown that the category of Markov processes on Polish spaces, whose morphisms are transition probability preserving, surjective continuous maps has semi-pullbacks. Any Polish space can be embedded in the subset of maximal elements of a ω-continuous dcpo ([Law97], [EH98], and Sections 5.3.4, 5.3.5). From the point of view of computability and computation it is important to determine whether regular conditional probabilities exist on continuous dcpo's and if their domain theoretic integrals can be used to approximate the regular conditional distributions on an embedded Hausdorff space (see also Sections 5 and 6 of [How00]). Theorem 4.27 is the first step in this direction but a lot of work remains to be done.

4.2 A review of perfect measures and projective limits of measure spaces

Perfect measures (also called quasi-compact measures) are a generalisation of compact measures and were originally introduced by Gnedenko and Kolmogorov [GK49]. The definition is as follows[1].

Definition 4.1 *Let (Ω, Γ, μ) be a measure space. Consider the measurable space $(\mathbb{R}, \mathcal{B}\mathbb{R})$. The measure μ is* ***perfect*** *if for each measurable function $f : \Omega \to \mathbb{R}$ and for each $E \subseteq \mathbb{R}$, such that $f^{-1}E \in \Gamma$, there exists a Borel set $B \in \mathcal{B}\mathbb{R}$ such that $B \subseteq E$ and $\mu(f^{-1}B) = \mu(f^{-1}E)$.*

We summarise some of the properties of perfect measures in the following proposition. Our main references in this section are [Fro72][Top79][Mus80] and [HJ94]. We use definitions and results from Section 3.2 extensively.

Proposition 4.2 *Let (Ω, Γ, μ) be a measure space. Then,*

1. *μ is perfect if and only if for each measurable function $f : \Omega \to \mathbb{R}$ there exists a Borel set $B \in \mathcal{B}\mathbb{R}$ such that $B \subseteq f[\Omega]$ and $\mu(f^{-1}B) = \mu(\Omega)$.*
2. *μ is perfect if and only if for each countably generated σ-algebra $\Sigma \subseteq \Gamma$ the measure $\mu \restriction_\Sigma$ is compact.*
3. *A finite measure μ is perfect if and only if for each sequence (E_n) of measurable sets and for each $\varepsilon > 0$ there exists a set $E \in \Gamma$ such that $\mu(\Omega) - \varepsilon < \mu(E)$ and the set $\{E \cap E_n\}_{n \in \mathbb{N}}$ is a semicompact paving.*
4. *μ is perfect if and only if its completion $(\Omega, \mathcal{M}(\mu), \widehat{\mu})$ is perfect.*
5. *If μ is perfect then, for any σ-algebra $\Sigma \subseteq \Gamma$ the measure $\mu \restriction_\Sigma$ is perfect.*
6. *If μ is perfect then, for any $E \in \Gamma$ the measure $(\Omega, E \cap \Gamma, \mu \restriction_{E \cap \Gamma})$ is*

[1] As most authors do, we do not require perfect measures to be complete departing from the original definition by Gnedenko and Kolmogorov.

perfect.

7. *The Cartesian product of perfect measures is perfect.*

8. *Every compact measure is perfect.*

Proof See [RN53] or Lemma 2, Theorems 1, 3, 2 and Section 1.6 in [Saz65]. ■

Next we turn our attention to projective systems of measure spaces which generalise Cartesian products of measure spaces. A quick overview of sheaf theory and pointers to the literature can be found in Chapter V of [Joh82], for more information on categorical projective limits see Section IV-3 in [GHK$^+$80]. The section on cylindrical measures in [Sch73] should also be mentioned.

Definition 4.3 *A* ***projective system of measure spaces*** *is a tuple* $(\Omega_i, \Gamma_i, \mu_i, p_{ji})_D$ *where*

1. $(D, \sqsubseteq)$ *is a directed set.*

2. $(\Omega_i, \Gamma_i, \mu_i)$ *is a measure space for all* $i \in D$.

3. *For all* $i, j \in D$ *with* $i \sqsubseteq j$ *we have that* $p_{ji} : \Omega_j \to \Omega_i$ *is a* (Γ_j, Γ_i)*-measurable function satisfying:*

   - p_{ii} *is the identity function on* Ω_i *for all* $i \in D$.

   - $p_{ki} = p_{ji} \circ p_{kj}$ *for all* $i \sqsubseteq j \sqsubseteq k$.

   - $\mu_j(p_{ji}^{-1} E) = \mu_i(E)$ *for all* $E \in \Gamma_i$ *and* $i \sqsubseteq j$. *That is, the functions* p_{ji} *are measure preserving.*

In other words, $(\{(\Omega_i, \Gamma_i, \mu_i)\}, \{p_{ji} \mid i \sqsubseteq j\})$ is a **presheaf** of measure spaces over $(D, \sqsubseteq)$. We say that (Ω_D, p_i) is a **target space** associated with $(\Omega_i, \Gamma_i, \mu_i, p_{ij})_D$ if

- $p_i : \Omega_D \to \Omega_i$ is a surjective map for all $i \in D$ (this condition could be relaxed to $p_i[\Omega_D]$ is a thick subset of $(\Omega_i, \Gamma_i, \mu_i)$; that is, $(\mu_i)_*(\Omega_i \backslash p_i[\Omega_D]) = 0$).

- $p_i = p_{ji} \circ p_j$ for all $i \sqsubseteq j$.

The most natural candidate for a target space is $\Omega_D = \{(\omega_i)_{i \in D} \in \Pi_i \Omega_i \mid p_{ji}(\omega_j) = \omega_i \text{ for all } i \sqsubseteq j\}$ and $p_i := \pi_i \restriction_{\Omega_D}$ for all $i \in D$, that is the restriction to Ω_D of the natural projection in the Cartesian product. Suppose that there exists[2] a target space (Ω_D, p_i) associated with $(\Omega_i, \Gamma_i, \mu_i, p_{ij})_D$. Then p_{ji} is surjective for all $i \sqsubseteq j$. Let $\mathcal{A}_D = \bigcup_{i\in D} \sigma(p_i) = \bigcup_{i\in D}\{p_i^{-1}E \mid E \in \Gamma_i\}$. Then $\mathcal{A}_D$ is the minimum algebra that makes all the functions p_i measurable. Define

$$\mu(p_i^{-1}E) := \mu_i(E) \text{ for all } E \in \Gamma_i.$$

Since D is directed, μ is a content on $(\Omega_D, \mathcal{A}_D)$ and we write $\mu = \varprojlim \mu_i$.

Definition 4.4 *We say that the projective system of measure spaces $(\Omega_i, \Gamma_i, \mu_i, p_{ij})_D$ is* ***convergent*** *if,*

- *There exists an associated target space (Ω_D, p_i).*
- *$\varprojlim \mu_i$ is σ-subadditive.*

Hence by Proposition 1.23-(2) $\varprojlim \mu_i$ extends uniquely to a measure μ_D on (Ω_D, Γ_D) where $\Gamma_D = \sigma(\mathcal{A}_D)$. We say that $(\Omega_D, \Gamma_D, \mu_D)$ is the ***projective limit*** *or* ***indirect product*** *of the system $(\Omega_i, \Gamma_i, \mu_i, p_{ij})_D$.*

Remark 4.5 *[Fro72] If there exists a target space (Ω_D, p_i) associated with $(\Omega_i, \Gamma_i, \mu_i, p_{ij})_D$ and every countable $B \subseteq D$ has an upper bound, then $\mathcal{A}_D$ is a σ-algebra and $\varprojlim \mu_i$ is a measure. Therefore $(\Omega_i, \Gamma_i, \mu_i, p_{ij})_D$ is convergent.*

Proposition 4.6 ***Projective Limit Theorem*** *(for compact measures)*

Suppose that $(\Omega_i, \Gamma_i, \mu_i, p_{ij})_D$ is a projective system of measure spaces where $\mu_i(\Omega_i) < \infty$ for all $i \in D$ and (Ω_D, p_i) is an associated target space. If there exists a paving $\mathcal{C}$ on Ω_D and pavings $\mathcal{C}_i$ on Ω_i for all $i \in D$ such that

1. *$\mathcal{C}$ is semicompact.*
2. *$\mathcal{C}_i$ is an approximating paving for μ_i.*
3. *$p_i^{-1}K \in \mathcal{C}$ for all $K \in \mathcal{C}_i$ and $i \in D$.*

Then $(\Omega_i, \Gamma_i, \mu_i, p_{ij})_D$ is convergent.

Proof See Section 1.50 in [HJ94]. ■

Notice that since p_i is onto, the third condition implies that $\mathcal{C}_i$ is semicompact and hence μ_i is a compact measure for all $i \in D$. It should be noted that Marczewski in his seminal paper [Mar53, Section 6], was the first one to obtain an abstract version of Kolmogorov's Theorem using compact measures.

[2]See Example 4.1 in [Mus80] for a system of measure spaces where the target space does not exist.

Many variations of this result had been given since requiring topological and other conditions from the measures and the σ-algebras. In particular, Musiał [Mus80] has given necessary and sufficient conditions for the existence of the projective limit when the measures are perfect (see Proposition 4.11 below). References to related work can be obtained from [Fro72][Mal75] and [Top79].

Corollary 4.7 *(Kolmogorov's Existence Theorem) Let T be any set and $(X_t, \mathcal{G}_t)_{t\in T}$ be a family of Hausdorff spaces. Suppose that $\mathbb{P}_A$ is a tight probability measure defined on $(\Pi_{t\in A}X_t, \otimes_{t\in A}\mathcal{B}X_t)$ for all $A \subseteq_f T$. If the family $\{\mathbb{P}_A\}_{A\subseteq_f T}$ is consistent, then there exists a probability measure $\mathbb{P}_T$ on $(\Pi_t X_t, \otimes_t \mathcal{B}X_t)$ such that $\mathbb{P}_A = \mathbb{P}_T \circ \pi_{TA}^{-1}$ for all $A \subseteq_f T$.*

Proof Let

- $D := 2_f^T$ ordered by $\subseteq$.
- $\Omega_A := \Pi_{t\in A}X_t$ for all $A \subseteq_f T$.
- $\Gamma_A := \otimes_{t\in A}\mathcal{B}X_t$ for all $A \subseteq_f T$.
- $p_{AB} := \pi_{AB}$ for all $B \subseteq A \subseteq_f T$.
- $\Omega_D := \Pi_t X_t$.
- $p_A := \pi_{TA} : \Omega_D \to \Omega_A$.
- $\mathcal{C}_A := \mathcal{K}(\Pi_{t\in A}X_t)$. That is, $\mathcal{C}_A$ is the paving of compact subsets of $\Pi_{t\in A}X_t$ with the product topology $\otimes_{t\in A}\mathcal{G}_t$.
- $\mathcal{C} := \{\pi_A^{-1}K \mid K \in \mathcal{C}_A,\, A \subseteq_f T\}$.

Everything seems to be in place, except for the fact that we need to verify that $\mathcal{C}$ is a semicompact paving. In fact, showing this is the most difficult part in the direct proof of Kolmogorov's Theorem. Notice that the sets in $\mathcal{C}$ are of the form $K \times \Pi_{t\in T\setminus A}X_t$ with $K \in \mathcal{K}(\Pi_{t\in A}X_t)$, hence by Tychonoff's Theorem (Proposition 1.7) they will not be compact unless the X_t are compact. Surprisingly, $\mathcal{C}$ is still a semicompact paving. Let $(\langle K_n\rangle)$ be a sequence in $\mathcal{C}$. Then

$$\langle K_n\rangle = \pi_{A_n}^{-1}K_n$$

where A_n is a finite subset of T and $K_n \in \mathcal{K}(\Pi_{t\in A_n}X_t)$ for all $n \in \mathbb{N}$. We use a diagonal argument similar to the one in [Bil86, Theorem 36.1] to show that if $\bigcap_{n=0}^m \langle K_m\rangle \neq \emptyset$ for all $m \in \mathbb{N}$, then $\bigcap_{n\in\mathbb{N}}\langle K_m\rangle \neq \emptyset$. Let $L_m := \bigcap_{n=0}^m \langle K_n\rangle$ and $A := \bigcup_{m\in\mathbb{N}} A_m$. For simplicity we denote the members of A_m as $\{(m,1),(m,2),...,(m,j_m)\}$ instead of $\{t_{(m,1)},...,t_{(m,j_m)}\}$ for all m. Then

$$L_m = \{(\omega_t) \in \prod_{t\in T} X_t \mid (\omega_{(n,1)}, \omega_{(n,2)}, \ldots, \omega_{(n,j_n)}) \in K_n \text{ for } n = 0, \ldots, m\}$$

Since we are assuming that L_m is nonempty, let $(\omega_t^{(m)})$ be a fixed element of L_m for each $m \in \mathbb{N}$. Notice that the set $\{(\omega_{(0,1)}^{(m)}, \omega_{(0,2)}^{(m)},...,\omega_{(0,j_0)}^{(m)}) \mid m \in \mathbb{N}\}$ is a sequence in the compact set K_0. Since $\Pi_{t\in A_0} X_t$ is a Hausdorff space, then K_0 is closed and there is a subsequence that converges to a point $(\omega_{(0,1)}^{\infty}, \omega_{(0,2)}^{\infty},...,\omega_{(0,j_0)}^{\infty})$ in K_0. Now assume that there exists a sequence (x_n) in $\mathbb{N}$ such that for each $r = 0,...,m$ the sequence $\{(\omega_{(r,1)}^{(x_n)}, \omega_{(r,2)}^{(x_n)},...,\omega_{(r,j_r)}^{(x_n)}) \mid n \in \mathbb{N}\}$ in K_r converges to a point $(\omega_{(r,1)}^{\infty}, \omega_{(r,2)}^{\infty},...,\omega_{(r,j_r)}^{\infty}) \in K_r$. Without loss of generality we can assume that $x_0 \geq m$ and $x_n \geq n$ for all $n \in \mathbb{N}$. Repeat the argument to obtain a subsequence (y_n) of $\{x_n \mid x_n \geq m+1\}$ such that $\{(\omega_{(m+1,1)}^{(y_n)}, \omega_{(m+1,2)}^{(y_n)},...,\omega_{(m+1,j_{m+1})}^{(y_n)}) \mid n \in \mathbb{N}\}$ converges to a point in K_{m+1}.

The set A_{m+1} need not be disjoint from the sets A_r with $0 \leq r \leq m$. Nevertheless, notice that for any subsequence (y_n) of (x_n), if $(m+1,j) = (r,k)$ with $(r,k) \in A_r$ for some $0 \leq r \leq m$, then $\lim_n \omega_{(m+1,j)}^{(y_n)} = \omega_{(r,k)}^{\infty}$; since in Hausdorff spaces any sequence has at most one limit point. In this way we have recursively defined a set $\{\omega_t^{\infty} \mid t \in A\}$. Choose any point $(\omega_t) \in \Pi_t X_t$ such that $\omega_t = \omega_t^{\infty}$ for every $t \in A$. Since $(\omega_{(m,1)}^{\infty}, \omega_{(m,2)}^{\infty},...,\omega_{(m,j_m)}^{\infty}) \in K_m$, then $(\omega_t) \in \langle K_m \rangle$ for all $m \in \mathbb{N}$, hence $\bigcap_{n\in\mathbb{N}} \langle K_m \rangle \neq \emptyset$. The conclusion is immediate from the definitions and the previous proposition. ■

4.3 Perfectness and monocompact contents

Suppose that Ω is a set and $\mathcal{A}$ is an algebra of subsets of Ω. Recall from Section 3.2 that $\mu : \mathcal{A} \to \mathbb{R}_+$ is a compact content if there exists a semicompact paving $\mathcal{C}$ approximating μ on $\mathcal{A}$. Now suppose that $\mathcal{S}$ is a semialgebra on Ω and $\eta : \mathcal{S} \to \mathbb{R}_+$ is a finitely additive set function. Since $\mathcal{A}(\mathcal{S})$ consists of finite disjoint unions of elements of $\mathcal{S}$, it follows that η extends uniquely to a content $\overline{\eta}$ on $\mathcal{A}(\mathcal{S})$. If there exists a semicompact paving $\mathcal{C}$ approximating η on $\mathcal{S}$, then the $(\cup f)$-closure of $\mathcal{C}$ is a semicompact paving (see Section 3.2) which is easily seen to approximate $\overline{\eta}$ on $\mathcal{A}(\mathcal{S})$. Therefore η can be extended uniquely to a compact measure on $\sigma(\mathcal{S})$ (Proposition 3.2). Consider the following definition and result.

Definition 4.8 *[Top79] Suppose that $\mathcal{A}$ is an algebra on Ω. A content $\eta : \mathcal{A} \to \mathbb{R}_+$ is **monocompact** if there is a monocompact paving $\mathcal{P}$ approx-*

imating η on Ω.

Proposition 4.9 *Suppose that $\mathcal{A}$ is an algebra on Ω and $\mu : \mathcal{A} \to \mathbb{R}_+$ is a monocompact content. Then,*

1. *(Mallory) μ can be extended uniquely to a measure $\overline{\mu}$ on $\sigma(\mathcal{A})$.*
2. *(Topsøe) The measure $\overline{\mu}$ is perfect.*

Proof For part (1) see Theorem 1.2 in [Mal75] (this also follows from Proposition 3.5). For part (2) see Theorem 2 in [Top79]. ■

In fact, the original result by Mallory was stated for semi-rings (semialgebras are particular instances of semi-rings), hence if $\eta : \mathcal{S} \to \mathbb{R}_+$ is as above and there exists a monocompact paving $\mathcal{P}$ approximating η on $\mathcal{S}$, then η can be extended to a measure $\overline{\eta}$ on $\sigma(\mathcal{S})$. In this setting is seems very natural to conjecture that $\overline{\eta}$ will also be a perfect measure. But in contrast to what happens with semicompact pavings, the $(\cup f)$-closure of $\mathcal{P}$ is not necessarily a semicompact or a monocompact paving. Even the set of finite disjoint unions of sets in $\mathcal{P}$ might not be a monocompact paving. Hence we do not know whether $\overline{\eta}\restriction_{\mathcal{A}(\mathcal{S})}$ is a monocompact content and cannot apply the previous proposition. The rest of this section is devoted to show that $\overline{\eta}$ is indeed perfect. We modify the proof of Theorem 2 in [Top79] for this setting. We will need one of Musial's characterisations of perfect measures via projective limits and some technical definitions.

Let Γ be a σ-algebra on Ω. A set $\Delta \in \Gamma$ is a **Γ-atom** if it is $\subseteq$-minimal in $\Gamma \backslash \{\emptyset\}$. That is, for all $F \in \Gamma$, $F \subseteq \Delta$ implies that $F = \emptyset$ or $F = \Delta$. We denote the set of atoms of Γ as $atm(\Gamma)$. Notice that if Γ is a finite algebra, then $atm(\Gamma)$ is a finite partition of Ω. A σ-algebra Γ is **quasi-countably generated** if there exists a countably generated sub-σ-algebra $\Gamma_0 \subseteq \Gamma$ such that $atm(\Gamma) = atm(\Gamma_0)$. Suppose that $(\Omega_i, \Gamma_i, \mu_i, p_{ij})_D$ is a projective system of measure spaces. If $J \subseteq D$ is also directed, we say that a family $\{E_i \in \Gamma_i\}_{i \in J}$ is **descending** if $E_j \subseteq p_{ji}^{-1}(E_i)$ for all $i \sqsubseteq j$.

Definition 4.10 *[Mus80, Definition 4.7]Let $(\Omega_i, \Gamma_i, \mu_i, p_{ij})_D$ be a projective system of countably generated measure spaces with associated target space (Ω_D, p_i). We say that the system is **atomic sequentially almost maximal** (a.s.a.m.) if, for every $\varepsilon > 0$ and every strictly increasing sequence (i_n) in D, there exists a sequence of sets $E_n \in \Gamma_{i_n}$ with the following properties (for all $n \in \mathbb{N}$):*

1. *(E_n) is descending. That is, $E_{n+1} \subseteq p_{i_{n+1}i_n}^{-1}(E_n)$.*

2. *$E_n \cap \Gamma_{i_n}$ is quasi-countably generated.*

3. *$\mu_{i_n}(\Omega_{i_n} \backslash E_n) < \varepsilon$.*

4. *For each descending sequence (Δ_m) of Γ_{i_m}-atoms with $\Delta_m \subseteq E_m$, there exists a point $x \in \Omega_D$ such that $p_{i_m}(x) \in \Delta_m$ for all $m \in \mathbb{N}$.*

Proposition 4.11 *(Musiał)* ***Projective Limit Theorem*** *(for countably generated perfect measure spaces)*

Let $(\Omega_i, \Gamma_i, \mu_i, p_{ij})_D$ be a projective system of measure spaces. If

1. *Γ_i is countably generated for all $i \in D$;*

2. *$(\Omega_i, \Gamma_i, \mu_i)$ is a perfect measure space for all $i \in D$;*

3. *$(\Omega_i, \Gamma_i, \mu_i, p_{ij})_D$ is a.s.a.m;*

then $(\Omega_D, \Gamma_D, \mu_D)$ exists (that is, the system is convergent) and is perfect.

Proof See Theorem 7.1 in [Mus80]. ■

Lemma 4.12 *Let Γ be an algebra on Ω. A measure space $(\Omega, \sigma(\Gamma), \mu)$ with $\mu(\Omega) < \infty$ is perfect if for every $\varepsilon > 0$ and every sequence (Γ_n) of finite subalgebras of Γ such that $\Gamma_n \subseteq \Gamma_{n+1}$ for all n, we can extract a sequence $E_n \in \Gamma_n$ with the following properties:*

1. *$E_{n+1} \subseteq E_n$ for all $n \in \mathbb{N}$.*

2. *$\mu(\Omega \backslash E_n) < \varepsilon$ for all $n \in \mathbb{N}$.*

3. *For each decreasing sequence (Δ_n) of Γ_n-atoms with $\Delta_n \subseteq E_n$, we have that $\bigcap_{n \in \mathbb{N}} \Delta_n \neq \emptyset$.*

Proof Define the projective system of measures spaces $(\Omega_i, \Gamma_i, \mu_i, p_{ij})_D$ as,

- $D := 2^{\Gamma}_f$, that is, the paving of finite subsets of Γ, ordered by $\subseteq$.
- For every $i \in D$ let $\Gamma_i := \mathcal{A}(i \cup \{\emptyset\})$. Then $\{\Gamma_i\}_{i \in D}$ is the family of all finite subalgebras of Γ ordered by inclusion.
- Let $\mu_i := \mu \restriction_{\Gamma_i}$and $\Omega_i := \Omega$ for all $i \in D$.
- Define p_{ji} as the identity map $Id_\Omega : \Omega \to \Omega$ for all $i \sqsubseteq j$. Then $p_{ji} : (\Omega, \Gamma_j) \to (\Omega, \Gamma_i)$ is clearly (Γ_j, Γ_i)-measurable since $i \sqsubseteq j$ implies that $\Gamma_i = \mathcal{A}(i \cup \{\emptyset\}) \subseteq \mathcal{A}(j \cup \{\emptyset\}) = \Gamma_j$.

Obviously (Ω, Id_Ω) is a target space associated with $(\Omega, \Gamma_i, \mu \restriction_{\Gamma_i}, Id_\Omega)_D$. It follows that $\mathcal{A}_D = \bigcup^{\uparrow}_{i \in D} \Gamma_i = \Gamma$ and $\varprojlim (\mu \restriction_{\Gamma_i}) = \mu \restriction_\Gamma$. Notice as well that for all $i \in D$ we have that,

1. Γ_i is countably generated since it is finite.
2. $(\Omega, \Gamma_i, \mu \restriction_{\Gamma_i})$ is perfect, moreover compact, since $\mu \restriction_{\Gamma_i}$ is a content on the finite algebra Γ_i.

It follows from Proposition 4.11 that $\mu = \mu_D$ is a perfect measure on $\sigma(\Gamma) = \sigma(\mathcal{A}_D)$ if $(\Omega, \Gamma_i, \mu \restriction_{\Gamma_i}, Id_\Omega)_D$ is a.s.a.m.; which, unwinding the definitions, is exactly what we require in the statement of this lemma. ■

Now we can formulate the main result of the chapter. The set of **multi-indices** is defined as $\mathbb{N}^{(\mathbb{N})} := \bigcup_{n\in\mathbb{N}} \mathbb{N}^n$ with $\mathbb{N}^0 = \{()\}$. That is $\mathbb{N}^{(\mathbb{N})} = \{(n_1, n_2,..., n_k) \in \mathbb{N}^k \mid k \in \mathbb{N}_+\} \cup \{()\}$. A **Souslin scheme** [HJ94, Exercise 1.91] on a set Ω is a function $\Phi : \mathbb{N}^{(\mathbb{N})} \to 2^\Omega$.

Theorem 4.13 *Let $\mathcal{S}$ be a semialgebra on Ω. Suppose that $\eta : \mathcal{S} \to \mathbb{R}_+$ is a finitely additive set function and there exists a monocompact paving $\mathcal{C}$ approximating η on $\mathcal{S}$. Let $\overline{\eta}$ denote the extension of η to a measure on $\sigma(\mathcal{S})$. Then $\overline{\eta}$ is perfect.*

Proof (Similar argument as Theorem 2 in [Top79]) We claim that for any finite subset $F \subseteq_f \mathcal{A}(\mathcal{S})$ there exists a finite set $[F] \subseteq \mathcal{S}$ (not necessarily unique) such that: (i) The elements of $[F]$ are pairwise disjoint. (ii) For each $e \in F$ we have that $e = \sum\{S \in [F] \mid S \subseteq e\}$. This can be done as follows. Notice that for each $e \in F$, $e = \sum\{\Delta \in atm(\mathcal{A}(F)) \mid \Delta \subseteq e\}$. In turn, each $\Delta \in atm(\mathcal{A}(F))$ can be expressed as a finite disjoint union $\sum_{i=1}^n S_i^\Delta$ where $S_i^\Delta \in \mathcal{S}$ (this expression might not be unique). Let $[F] := \{S_i^\Delta \in \mathcal{S} \mid \Delta \in atm(\mathcal{A}(F))\}$.

Let $(\Gamma_n)_{n\geq 1}$ be any increasing sequence of finite subalgebras of $\mathcal{A}(\mathcal{S})$ and $\varepsilon > 0$. We define Souslin schemes $S(\_)$ and $K(\_)$ on Ω satisfying the following conditions for all $(n_1,..., n_k) \in \mathbb{N}^{(\mathbb{N})}$:

1. $S(n_1,..., n_k) \in \mathcal{S}$.
2. Fix $k \in \mathbb{N}_+$. Then $S(n_1,..., n_k) = \emptyset$ for all but a finite number of multi-indices in $\mathbb{N}^k$ and the sets $S(n_1,..., n_k)$ are disjoint.
3. There exists exactly one $\Delta \in atm(\Gamma_k)$ such that $S(n_1,..., n_k) \subseteq \Delta$.
4. $K(n_1,..., n_k) \in \mathcal{C}$ and
$$S(n_1,..., n_k, m) \subseteq K(n_1,..., n_k) \subseteq S(n_1,..., n_k)$$
for all $m \in \mathbb{N}$.

5. For each $k \in \mathbb{N}_+$ we have that

$$\mu(\sum_{(n_1,\ldots,n_k)\in\mathbb{N}^k} S(n_1,\ldots,n_k)) - \tfrac{\varepsilon}{2^k}$$
$$< \mu(\sum_{(n_1,\ldots,n_{k+1})\in\mathbb{N}^{k+1}} S(n_1,\ldots,n_{k+1})).$$

Remark The reader should observe that $S(\_)$ is just a finitely branching tree whose nodes are the sets $S(n_1,\ldots,n_k)$. Since $K(\_)$ is a monocompact paving interpolating $S(\_)$, it should be clear why if the intersection of an infinite branch is empty, then there must exist a finite sub-branch with empty intersection.

The recursive definition of $S(\_)$ and $K(\_)$ is as follows.

- **Base case.** Since $[\Gamma_1] = \{S_0,\ldots,S_n\}$ is a finite set, let $S(k) := S_k$ for $k = 0,\ldots,n$ and $S(k) = \emptyset$ otherwise. Notice that $\mu(\sum_{n_1\in\mathbb{N}} S(n_1)) = \mu(\Omega)$ since $[\Gamma_1]$ is a partition of Ω. For all $\Delta \in atm(\Gamma_1)$ we have that $\Delta = \sum\{S(n_1) \subseteq \Delta \mid n_1 \in \mathbb{N}\}$ hence conditions (1)-(3) are satisfied.

- **Recursive step.** Suppose that $S(n_1,\ldots,n_k)$ satisfies the conditions (1)-(3) above. Since $\mathcal{C}$ approximates μ on $\mathcal{S}$ there exist $A(n_1,\ldots,n_k) \in \mathcal{S}$ and $K(n_1,\ldots,n_k) \in \mathcal{C}$ such that $A(n_1,\ldots,n_k) \subseteq K(n_1,\ldots,n_k) \subseteq S(n_1,\ldots,n_k)$ with

  $$\mu(\sum_{(n_1,\ldots,n_k)\in\mathbb{N}^k} S(n_1,\ldots,n_k)) - \tfrac{\varepsilon}{2^k} < \mu(\sum_{(n_1,\ldots,n_k)\in\mathbb{N}^k} A(n_1,\ldots,n_k)).$$

  Define an auxiliary set F as

  $$F := \Gamma_{k+1} \cup \{A(n_1,\ldots,n_k) \mid (n_1,\ldots,n_k) \in \mathbb{N}^k\}.$$

  By condition (2) F is a finite set. Now rename the members of $\{S \in [F] \mid S \subseteq A(n_1,\ldots,n_k)\}$ to be the sets $S(n_1,\ldots,n_k,m)$ with m ranging over all $\mathbb{N}$ (as in the base case). Clearly conditions (1)-(4) hold. For verifying condition (5) we just have to notice that $A(n_1,\ldots,n_k) = \sum_{m\in\mathbb{N}} S(n_1,\ldots,n_k,m)$.

For all $k \in \mathbb{N}_+$, let

$$\mathcal{E}_k := \{\Delta \in atm(\Gamma_k) \mid \text{there exists } \emptyset \neq S(n_1,\ldots,n_k) \subseteq \Delta\}$$

Define the sequence of sets $E_k \in \Gamma_k$ as $E_k := \bigcup \mathcal{E}_k$. We check that the hypotheses in Lemma 4.12 are satisfied.

(a) **Claim:** $E_{k+1} \subseteq E_k$ for all $k \in \mathbb{N}_+$. Suppose that $\Delta_{k+1} \in \mathcal{E}_{k+1}$. Then there exists a multi-index $(n_1,\ldots,n_k,n_{k+1})$ such that $\emptyset \neq S(n_1,\ldots,n_k,$

$n_{k+1}) \subseteq \Delta_{k+1}$. Since $\Gamma_k \subseteq \Gamma_{k+1}$, then there exists exactly one $\Delta_k \in atm(\Gamma_k)$ such that $\Delta_{k+1} \subseteq \Delta_k$ (could be Δ_{k+1} itself). But by conditions (3) and (4) we have that $S(n_1,...,n_k,n_{k+1}) \subseteq S(n_1,...,n_k)$ and $S(n_1,...,n_k)$ is contained in exactly one atom of Γ_k. Since the atoms of Γ_k are disjoint it follows that $S(n_1,...,n_k) \subseteq \Delta_k$. Therefore for all $\Delta_{k+1} \in \mathcal{E}_{k+1}$ there exists $\Delta_k \in \mathcal{E}_k$ with $\Delta_{k+1} \subseteq \Delta_k$. It follows that $E_{k+1} \subseteq E_k$.

(b) **Claim:** $\mu(\Omega \backslash E_k) < \varepsilon$ for all $k \in \mathbb{N}_+$. Notice that:

$$\begin{aligned} \mu(\Omega) - \varepsilon &= \mu(\textstyle\sum_{n_1 \in \mathbb{N}} S(n_1)) - \varepsilon && \text{by the base case} \\ &< \mu(\sum_{(n_1,...,n_k) \in \mathbb{N}^k} S(n_1,...,n_k)) && \text{by condition (5)} \\ &\leq \mu(E_k) && \text{by condition (3)} \end{aligned}$$

(c) **Claim:** For each decreasing sequence $(\Delta_k)_{k \geq 1}$ of Γ_k-atoms with $\Delta_k \subseteq E_k$, we have that $\bigcap_{k \geq 1} \Delta_k \neq \emptyset$. Notice that $\Delta_k \in \mathcal{E}_k$ and $\Delta_{k+1} \subseteq \Delta_k$ for all $k \in \mathbb{N}_+$. From the proof of claim (a) it is clear that if $S(n_1,...,n_k,n_{k+1}) \subseteq \Delta_{k+1}$ then $S(n_1,...,n_k) \subseteq \Delta_k$. From condition (4) we know that

$$S(n_1,...,n_k,n_{k+1}) \subseteq K(n_1,...,n_k) \subseteq S(n_1,...,n_k)$$

Hence there exists a sequence (n_k) of natural numbers such that

$$\begin{aligned} &K(n_1,...,n_k) \subseteq \Delta_k \\ &K(n_1,...,n_k,n_{k+1}) \subseteq K(n_1,...,n_k) \end{aligned}$$

for all $k \in \mathbb{N}$. Therefore $\bigcap_{k \geq 1} K(n_1,...,n_k) \subseteq \bigcap_{k \geq 1} \Delta_k$. But the first intersection cannot be empty since it is a decreasing sequence in the monocompact paving $\mathcal{C}$.

By Lemma 4.12 we conclude that $(\Omega, \sigma(\mathcal{A}(\mathcal{S})), \overline{\eta}) = (\Omega, \sigma(\mathcal{S}), \overline{\eta})$ is a perfect measure space. ■

Remark 4.14

- *We still do not know if $\overline{\eta}$ is a monocompact content on $\mathcal{A}(\mathcal{S})$. Topsøe [Top79] has defined a weaker notion called asymptotical monocompactness (which we will not discuss here). The most we can say is that $\overline{\eta}$ is asymptotically monocompact on $\mathcal{A}(\mathcal{S})$.*

- *In the same way as Proposition 3.5 generalises Mallory's result [Mal75, Theorem 1.2] for semi-rings, it should be possible to show that the extension given by Proposition 3.5 is a perfect measure, but we are not interested in pursuing such a result.*

Recall the statement of Theorem 3.27.

Corollary 4.15 ***Perfectness of finite valuations on locally compact sober spaces***

Let $(X, \mathcal{G})$ be a locally compact sober space and let ν be a finite continuous valuation on X. Let $\overline{\nu}$ denote the unique extension of ν to a τ-smooth measure on $\mathcal{B}X$. Then the measure space $(X, \mathcal{B}X, \overline{\nu})$ is perfect.

Proof Recall from Section 1.11 that $Cres(X)$ is a semialgebra with $\sigma(Cres(X)) = \mathcal{B}X$. By the proof of Theorem 3.27 we know that $Lens(X)$ is a mono-compact paving approximating $\overline{\nu}$ on $Cres(X)$. ■

From Proposition 1.10 we obtain:

Corollary 4.16 ***Perfectness of finite valuations on continuous dcpo's***

Let D be a continuous dcpo and let ν be a finite continuous valuation $(D, \mathcal{G}_\sigma D)$. Let $\overline{\nu}$ denote the unique extension of ν to a τ-smooth Borel measure on $\mathcal{B}_\sigma D$. Then the measure space $(D, \mathcal{B}_\sigma D, \overline{\nu})$ is perfect.

Suppose that $(X, \mathcal{G})$ is a Hausdorff topological space and μ is a finite measure on $(X, \mathcal{B}X)$. Recall from Section 1.7 that if μ is inner regular, then μ is τ-smooth and $\mathcal{K}$-compact, hence perfect by Proposition 4.2-(8). On the other hand there exist examples of perfect measures that are not τ-smooth and τ-smooth measures that are not perfect [Saz65, Section 2].

Problem 4.17 *Let $(X, \mathcal{G})$ be a locally compact sober space and let ν be a finite continuous valuation on X. Let $\overline{\nu}$ denote the unique extension of ν to a τ-smooth measure on $\mathcal{B}X$. Is the measure $\overline{\nu}$ compact? Give an example where $\overline{\nu}$ is (perfect but) not compact.*

4.4 Factorization, disintegration and regular conditional probability distributions

In the following definitions (X_1, Σ_1) and (X_2, Σ_2) denote measurable spaces. Recall that $\Sigma_1 \otimes \Sigma_2 = \sigma(\{E \times F \mid E \in \Sigma_1 \text{ and } F \in \Sigma_2\})$. If $\mathbb{P}$ is a probability defined on the measurable space $(X_1 \times X_2, \Sigma_1 \otimes \Sigma_2)$, then $\pi_i[\mathbb{P}]$ denotes the marginal distributions of $\mathbb{P}$. That is, $\pi_i[\mathbb{P}](E) = \mathbb{P}_{\pi_i}(E) = \mathbb{P}(\pi_i^{-1}E)$ for all $E \in \Sigma_i$ and $i = 1, 2$. Conversely, if $\mathbb{P}_1$ and $\mathbb{P}_2$ are probabilities on (X_1, Σ_1) and (X_2, Σ_2) respectively, then a probability $\mathbb{P}$ on $(X_1 \times X_2, \Sigma_1 \otimes \Sigma_2)$ is a joint probability if $\mathbb{P}_i = \pi_i[\mathbb{P}]$ for $i = 1, 2$.

Suppose that $\mathbb{P}$ is a probability on $(X_1 \times X_2, \Sigma_1 \otimes \Sigma_2)$. Let $\mathbb{P}_2 := \pi_2[\mathbb{P}]$.

A family $(\Sigma_1^y, \mathbb{P}_1^2(\_ \mid y))_{y\in X_2}$ is called a **weak $\mathbb{P}_2$-disintegration** [Pac78, Definition 1.1] of $\mathbb{P}$ if

1. $\mathbb{P}_1^2(\_ \mid y)$ is a probability measure on (X_1, Σ_1^y) for all $y \in X_2$.
2. Fix $E \in \Sigma_1$. Then there exists a set $N_E \in \Sigma_2$ such that
   - $\mathbb{P}_2(N_E) = 0$.
   - $E \in \Sigma_1^y$ for all $y \in X_2 \backslash N_E$. ([3])
   - The function $\mathbb{P}_1^2(E \mid \_) : X_2\backslash N_E \to [0,1]$ is $(\Sigma_2 \restriction_{X_2\backslash N_E})$-measurable.
3. For all $E \in \Sigma_1$ and $F \in \Sigma_2$
$$\mathbb{P}(E \times F) = \int_F \mathbb{P}_1^2(E \mid y)\mathbb{P}_2(dy).$$

If $(\Sigma_1^y, \mathbb{P}_1^2(\_ \mid y))_{y\in X_2}$ is a weak $\mathbb{P}_2$-disintegration of $\mathbb{P}$, then $(\Sigma_1 \cap \Sigma_1^y, \mathbb{P}_1^2(\_ \mid y))_{y\in X_2}$ is also a weak $\mathbb{P}_2$-disintegration of $\mathbb{P}$. Hence, without loss of generality, we can assume that $\Sigma_1^y \subseteq \Sigma_1$ for all $y \in X_2$. If $\Sigma_1^y = \Sigma_1$ for all $y \in X_2$, then $N_E = \emptyset$ for all $E \in \Sigma_1$ and we have the following definition.

Definition 4.18 *Let (X_1, Σ_1) and (X_2, Σ_2) be measurable spaces and $\mathbb{P}$ be a probability defined on $(X_1 \times X_2, \Sigma_1 \otimes \Sigma_2)$. Let $\mathbb{P}_2 := \pi_2[\mathbb{P}]$. We say that $\mathbb{P}_1^2(\_ \mid y)_{y\in X_2}$ is a $\mathbb{P}_2$-**disintegration**[4] of $\mathbb{P}$ if*

1. *$\mathbb{P}_1^2(\_ \mid y)$ is a probability measure on (X_1, Σ_1) for all $y \in X_2$.*
2. *$\mathbb{P}_1^2(E \mid \_) : X_2 \to [0,1]$ is a Σ_2-measurable function for all $E \in \Sigma_1$.*
3. *For all $E \in \Sigma_1$ and $F \in \Sigma_2$*
$$\mathbb{P}(E \times F) = \int_F \mathbb{P}_1^2(E \mid y)\mathbb{P}_2(dy).$$

Our definition of weak disintegration corresponds to Pachl's definition of disintegration. Pachl wants the measure $\mathbb{P}(B) := \mathbb{P}_1(\{(x,y) \in [0,1]^2 \mid ((x,y),z) \in B\})$ to have a $\mathbb{P}_2$-disintegration where $X_1 = [0,1]^2$, $X_2 = [0,1]$ and $\mathbb{P}_i$ are the complete Lebesgue probability measures on X_i. If this were true for our definition, we would contradict the continuum hypothesis (see Example 1.2 in [Pac78] and Section 10.26 on [HJ94]). We, on the other hand, are interested in disintegrations that correspond to regular conditional probability distributions. Hence the converse of Proposition 4.21 below does not hold and the most we can say is in Proposition 4.22.

[3]Intuitively this means something like $\Sigma_1 \subseteq \Sigma_1^y$ for $\mathbb{P}_2$-almost all $y \in X_2$.
[4]Decent disintegration in [Pac78].

Remark 4.19 *Hoffmann-Jørgensen [HJ94, 10.30] defines disintegration in a different way; see Section 4.5 at the end of this chapter.*

Let $(\Omega, \Gamma, \mathbb{P})$ be a probability space and let (X_1, Σ_1) and (X_2, Σ_2) denote measurable spaces. Let $S : \Omega \to X_1, T : \Omega \to X_2$ be two random functions. Suppose that $\mathcal{D}$ is a sub σ-algebra of $\mathcal{M}(\mathbb{P}_T)$. A **$\mathcal{D}$-measurable** regular conditional probability distribution (rcpd) [Eda99, Section 2] of S given T is a function $\mathbb{P}_S^T(\_ \mid \_) : \Sigma_1 \times X_2 \to [0,1]$ satisfying:

1. $\mathbb{P}_S^T(\_ \mid y)$ is a probability measure on (X_1, Σ_1) for all $y \in X_2$.
2. $\mathbb{P}_S^T(E \mid \_) : X_2 \to [0,1]$ is a $\mathcal{D}$-measurable function for all $E \in \Sigma_1$.
3. For all $E \in \Sigma_1$ and $F \in \Sigma_2$

$$\mathbb{P}(S \in E, T \in F) = \int_F \mathbb{P}_S^T(E \mid y)\mathbb{P}_T(dy).$$

A Σ_2-measurable rcpd is simply called **regular conditional probability distribution** [HJ94, Vol 2, p. 117]. A $\mathcal{U}(\Sigma_2)$-measurable rcpd is called a **universally measurable** regular conditional probability distribution (um-rcpd). A $\mathcal{M}(\mathbb{P}_T)$-measurable rcpd is called an **almost surely** regular conditional probability distribution (as-rcpd) [HJ94, Vol 2, p. 152].

If $Q_S^T(\_ \mid \_)$ is any other $\mathcal{D}$-measurable rcpd of S given T then for all $E \in \Sigma_1$ we have that $Q_S^T(E \mid \_) = \mathbb{P}_S^T(E \mid \_)$ $\mathbb{P}_T$-almost everywhere in Σ_2. If $\mathbb{P}_T$ is complete, then $\Sigma_2 = \mathcal{M}(\mathbb{P}_T)$ and every as-rcpd is a rcpd [HJ94, 10.26.2]. Next we show that disintegration and rcpd are, in fact, equivalent concepts. Let $(X_1, \Sigma_1, \mathbb{P}_1)$ and $(X_2, \Sigma_2, \mathbb{P}_2)$ be probability spaces and suppose that $\mathbb{P}$ is a joint probability. Since $\mathbb{P}(\pi_1 \in E, \pi_2 \in F) = \mathbb{P}(E \times F)$, then

- $\mathbb{P}_1^2(\_ \mid y)_{y \in X_2}$ is a $\mathbb{P}_2$-disintegration of $\mathbb{P}$ if and only if $\mathbb{P}_1^2(\_ \mid \_)$ is a rcpd of π_1 given π_2.

Conversely, let $(\Omega, \Gamma, \mathbb{P})$ be a probability space and let (X_1, Σ_1) and (X_2, Σ_2) be measurable spaces. Suppose that $S : \Omega \to X_1, T : \Omega \to X_2$ are two random functions. Let the random vector $(S,T) : \Omega \to X_1 \times X_2$ be defined as $(S,T)(\omega) = (S(\omega), T(\omega))$. Since $\mathbb{P}_{(S,T)}(E \times F) = \mathbb{P}(S \in E, T \in F)$, then $(X_1, \Sigma_1, \mathbb{P}_S)$ and $(X_2, \Sigma_2, \mathbb{P}_T)$ are probability spaces and $\mathbb{P}_{(S,T)}$ is a joint probability. Hence,

- $\mathbb{P}_S^T(\_ \mid \_)$ is a rcpd of S given T if and only if $\mathbb{P}_S^T(\_ \mid y)_{y \in X_2}$ is a $\mathbb{P}_T$-disintegration of $\mathbb{P}_{(S,T)}$.

Suppose that $\mathcal{C}$ is an $(\cup f, \cap c)$-semicompact paving on a set Ω. Let $\widehat{\mathcal{C}}$ be the paving defined as

$$\widehat{\mathcal{C}} := \{E \subseteq \Omega \mid E \cap K \in \mathcal{C} \text{ for all } K \in \mathcal{C}\}$$

The following lemma gives sufficient conditions for the existence of rcpd's.

Lemma 4.20 *Let $(\Omega, \Gamma, \mathbb{P})$ be a probability space, (X_1, Σ_1) and (X_2, Σ_2) denote measurable spaces and $S : \Omega \to X_1, T : \Omega \to X_2$ be two random functions. Suppose that $\mathcal{C}$ is a $(\cup f, \cap c)$-semicompact paving on X_1 such that*

- $\mathcal{C} \subseteq \Sigma_1 \subseteq \sigma(\widehat{\mathcal{C}})$
- *$\mathbb{P}_S$ is a $\mathcal{C}$-compact measure on Σ_1.*

Then there exists an as-rcpd $\mathbb{P}_S^T(\_ \mid \_)$ of S given T such that $\mathbb{P}_S^T(\_ \mid y)$ is a $\mathcal{C}$-compact measure for all $y \in X_2$. Furthermore, if Σ_1 is countably generated, then there exists a rcpd $Q(\_ \mid \_)$ of S given T such that $Q(\_ \mid y) = \mathbb{P}_S^T(\_ \mid y)$ for $\mathbb{P}_T$-almost all $y \in \Sigma_2$.

Proof See Section 10.29 and Remark 10.26.3 in [HJ94]. ■

Proposition 4.21 *(Pachl) Let $(X_1, \Sigma_1, \mathbb{P}_1)$ be a probability space. Suppose that for every complete probability space $(X_2, \Sigma_2, \mathbb{P}_2)$, every joint probability $\mathbb{P}$ on $(X_1 \times X_2, \Sigma_1 \otimes \Sigma_2)$ has a $\mathbb{P}_2$-disintegration. Then $\mathbb{P}_1$ is a compact measure.*

Proof This follows from Definition 1.3 and Theorem 2.2 in [Pac78] since every $\mathbb{P}_2$-disintegration is, in particular, a weak $\mathbb{P}_2$-disintegration. ■

Proposition 4.22 *Let $(X_1, \Sigma_1, \mathbb{P}_1)$ and $(X_2, \Sigma_2, \mathbb{P}_2)$ be probability spaces. Suppose that $\mathcal{C}$ is an $(\cup f, \cap c)$-semicompact paving on X_1 such that*

- $\mathcal{C} \subseteq \Sigma_1 \subseteq \sigma(\widehat{\mathcal{C}})$
- *$\mathbb{P}_1$ is a $\mathcal{C}$-compact measure on Σ_1.*

Let $\mathbb{P}$ be a joint probability on $(X_1 \times X_2, \Sigma_1 \otimes \Sigma_2)$. Then,

1. *[Pac78, Theorem 3.5] If $(X_2, \Sigma_2, \mathbb{P}_2)$ is complete, then $\mathbb{P}$ has a $\mathbb{P}_2$-disintegration $\mathbb{P}_1^2(\_ \mid y)_{y \in X_2}$ such that $\mathbb{P}_1^2(\_ \mid y)$ is a $\mathcal{C}$-compact measure for all $y \in X_2$.*
2. *If Σ_1 is countably generated, then $\mathbb{P}$ has a $\mathbb{P}_2$-disintegration $\mathbb{P}_1^2(\_ \mid y)_{y \in X_2}$ such that $\mathbb{P}_1^2(\_ \mid y)$ is a $\mathcal{C}$-compact measure for $\mathbb{P}_T$-almost all $y \in X_2$.*

Proof Let $\mathbb{P}$ be a joint probability on $\Sigma_1 \otimes \Sigma_2$. By the previous lemma there exist an as-rcpd $\mathbb{P}_1^2(\_ \mid \_)$ of π_1 given π_2 such that $\mathbb{P}_1^2(\_ \mid y)$ is a $\mathcal{C}$-compact measure for all $y \in X_2$. If $(X_2, \Sigma_2, \mathbb{P}_2)$ is complete, then $\Sigma_2 = \mathcal{M}(\mathbb{P}_2)$ hence $\mathbb{P}_1^2(\_ \mid \_)$ is a rcpd. If Σ_1 is countably generated, then there exists a rcpd $Q(\_ \mid \_)$ of π_1 given π_2 and $Q(\_ \mid y) = \mathbb{P}_1^2(\_ \mid y)$ for $\mathbb{P}_T$-almost all $y \in \Sigma_2$. By the remarks above in both cases we obtain a $\mathbb{P}_2$-disintegration of $\mathbb{P}$ with the desired properties. ■

Remark 4.23 *In Section 4 of [Pac78] it is claimed that the first condition in Proposition 4.22 can be replaced by $\mathcal{C} \subseteq \Sigma_1 \subseteq \mathcal{U}(\mathcal{C})$. Where*

$$\mathcal{U}(\mathcal{C}) = \bigcap\{\mathcal{M}(\mathbb{P}) \mid \mathbb{P} \textit{is a } \mathcal{C}\textit{-compact probability measure on } (X_1, \sigma(\mathcal{C}))\}$$

It is possible to show (using the results in Section 1.31 of [HJ94] for example) that $\sigma(\widehat{\mathcal{C}}) \subseteq \mathcal{U}(\mathcal{C})$, but we do not know when these two pavings are the same. For practical purposes $\sigma(\widehat{\mathcal{C}})$ is 'big enough'.

Now we turn our attention to a weaker form of disintegration called factorization. Our main reference for this subject is Maitra and Ramakrishnan [MR88] (see also [BD75],[BM84] and [Ada86]).

Definition 4.24 *Let (X_1, Σ_1) and (X_2, Σ_2) be measurable spaces and $\mathbb{P}$ be a probability defined on $(X_1 \times X_2, \Sigma_1 \otimes \Sigma_2)$. Let $\mathbb{P}_2 := \pi_2[\mathbb{P}]$. We say that $\mathbb{F}_1^2(\_ \mid y)_{y \in X_2}$ is a Σ_2-**factorization** of $\mathbb{P}$. If*

1. *$\mathbb{F}_1^2(\_ \mid y)$ is a probability* content *on (X_1, Σ_1) for all $y \in X_2$.*

2. *For all $B \in \Sigma_1 \otimes \Sigma_2$ the map $\varphi_B : X_2 \to [0,1]$ given by $\varphi_B(y) := \mathbb{F}_1^2(B^y \mid y)$ is $\mathcal{M}(\mathbb{P}_2)$-measurable. Where $B^y := \{x \in X_1 \mid (x, y) \in B\}$ for all $y \in X_2$.*

3. *For all $B \in \Sigma_1 \otimes \Sigma_2$,*

$$\mathbb{P}(B) = \int \mathbb{F}_1^2(B^y \mid y)\mathbb{P}_2(dy).$$

In the previous definition notice that for $E \in \Sigma_1$ and $F \in \Sigma_2$ we have that

$$(E \times F)^y = \begin{cases} E & \text{if } y \in F \\ \emptyset & \text{otherwise} \end{cases}$$

In particular $(E \times X_2)^y = E$ for all $y \in X_2$. Hence conditions (2) and (3) above imply the more familiar conditions:

2′. $\mathbb{F}_1^2(E \mid \_) : X_2 \to [0,1]$ is a $\mathcal{M}(\mathbb{P}_2)$-measurable function for all $E \in \Sigma_1$.

3′. For all $E \in \Sigma_1$ and $F \in \Sigma_2$

$$\begin{aligned}\mathbb{P}(E \times F) &= \int \mathbb{I}_F \cdot \mathbb{F}_1^2(E \mid y)\mathbb{P}_2(dy)\\ &= \int_F \mathbb{F}_1^2(E \mid y)\mathbb{P}_2(dy)\end{aligned}$$

It is possible to show that the existence of a $\mathbb{P}_2$-disintegration implies the existence of a Σ_2-factorization (see remarks in Section 2 of [MR88]). The analogue of Propositions 4.21 and 4.22 are the following results.

Proposition 4.25 *Let $(X_1, \Sigma_1, \mathbb{P}_1)$ be a probability space. Suppose that for any probability space $(X_2, \Sigma_2, \mathbb{P}_2)$, every joint probability $\mathbb{P}$ on $(X_1 \times X_2, \Sigma_1 \otimes \Sigma_2)$ has a Σ_2-factorization. Then $\mathbb{P}_1$ is a perfect measure.*

Proof Follows from Theorems 1 and 3 in [MR88]. ■

Proposition 4.26 *[MR88, Theorem 3]Let $(X_1, \Sigma_1, \mathbb{P}_1)$ be a probability space and suppose that Σ_1 is $\aleph_1$-generated. Then the following are equivalent:*

1. *$\mathbb{P}_1$ is a perfect measure.*
2. *For any probability space $(X_2, \Sigma_2, \mathbb{P}_2)$, every joint probability $\mathbb{P}$ on $(X_1 \times X_2, \Sigma_1 \otimes \Sigma_2)$ has a Σ_2-factorization.*
3. *For any probability space $(X_2, \Sigma_2, \mathbb{P}_2)$ with Σ_2 countably generated, every joint probability $\mathbb{P}$ on $(X_1 \times X_2, \Sigma_1 \otimes \Sigma_2)$ has a Σ_2-factorization.*
4. *For any probability measure $\mathbb{P}_2$ on $(\mathbb{R}, \mathcal{B}\mathbb{R})$ every joint probability $\mathbb{P}$ on $(X_1 \times \mathbb{R}, \Sigma_1 \otimes \mathcal{B}\mathbb{R})$ has a $\mathcal{B}\mathbb{R}$-factorization.*

Moreover if Σ_1 is countably generated, then the previous conditions are equivalent to

- *For any probability space $(X_2, \Sigma_2, \mathbb{P}_2)$, every joint probability $\mathbb{P}$ on $(X_1 \times X_2, \Sigma_1 \otimes \Sigma_2)$ has a* countably additive *Σ_2-factorization.*

Having set all the necessary mathematical machinery we can present the existence results of factorization and disintegration on locally compact sober spaces and continuous dcpo's. Recall that if $(X, \mathcal{G})$ is a topological space, then $\mathcal{B}X = \sigma(\mathcal{G})$ and $\mathcal{B}_pX = \sigma(\mathcal{G}_p)$ denote the Borel σ-algebras corresponding to the space and the patch space; similarly if $(D, \sqsubseteq)$ is a continuous dcpo, then $\mathcal{B}_\sigma D$ and $\mathcal{B}_\lambda D$ denote the Borel σ-algebras corresponding to the Scott and Lawson topologies. The hypothesis in the following theorem make implicit use of the extension results (Theorem 3.27, Corollary 3.29, Corollary 3.40) and Corollary 4.15.

Theorem 4.27 ***Existence of disintegrations and factorizations on locally compact sober spaces****. Let $(X, \mathcal{G})$ be a locally compact sober space* *and $\nu \in \mathbf{P}^1 X$. Let $\overline{\nu}$ denote the unique extension of ν to a perfect τ-smooth probability measure on $(X, \mathcal{B}X)$. Let $(X_2, \Sigma_2, \mathbb{P}_2)$ be any probability space. If $\mathbb{P}$ is a joint probability measure on $\mathcal{B}X \otimes \Sigma_2$, then*

1. *If $\mathcal{B}X$ is $\aleph_1$-generated, then there exists a Σ_2-factorization of $\mathbb{P}$.*

Moreover suppose that X is stably locally compact or second countable hence $\overline{\nu}$ can be assumed to be a Radon measure on $(X, \mathcal{B}_p X)$. If $\mathbb{P}$ is any joint probability measure on $\mathcal{B}_p X \otimes \Sigma_2$. Then,

2. *If $(X_2, \Sigma_2, \mathbb{P}_2)$ is complete, then there exists a $\mathbb{P}_2$-disintegration $\mathbb{P}^2_1(\_ \mid y)_{y \in X_2}$ of $\mathbb{P}$.*

3. *If X is second countable, then there exists a $\mathbb{P}_2$-disintegration $\mathbb{P}^2_1(\_ \mid y)_{y \in X_2}$ of $\mathbb{P}$.*

In both cases $\mathbb{P}^2_1(\_ \mid y)$ is a Radon measure for all $y \in X_2$. Hence the restriction of $\mathbb{P}^2_1(\_ \mid y)$ to $\mathcal{G}$ is a continuous valuation on X for all $y \in X_2$.

Proof Part (1) follows immediately from the equivalence (1)$\Leftrightarrow$(2) in Proposition 4.26. For parts (2) and (3) notice that $\overline{\nu}$ is a Radon measure if and only if $\overline{\nu}$ is $\mathcal{K}_p$-compact. By part (4) of Proposition 3.10 we know that $(X, \mathcal{G}_p)$ is a Hausdorff space, hence $\mathcal{K}_p \subseteq \mathcal{F}_p$ and $F \cap K \in \mathcal{K}_p$ for all $K \in \mathcal{K}_p$ and $F \in \mathcal{F}_p$. It follows that $\sigma(\mathcal{K}_p) \subseteq \sigma(\mathcal{F}_p)$ and $\mathcal{F}_p \subseteq \widehat{\mathcal{K}}_p$. Therefore

$$\sigma(\mathcal{K}_p) \subseteq \sigma(\mathcal{F}_p) = \mathcal{B}_p X \subseteq \sigma(\widehat{\mathcal{K}}_p),$$

and we can apply Proposition 4.22 to obtain a $\mathbb{P}_2$-disintegration $\mathbb{P}^2_1(\_ \mid y)_{y \in X_2}$ of $\mathbb{P}$. For case (2) $\mathbb{P}^2_1(\_ \mid y)_{y \in X_2}$ is guaranteed to be a Radon ($\mathcal{K}_p$-compact) measure for all $y \in X_2$. For case (3), $(X, \mathcal{G}_p)$ is a Polish space (Proposition 3.10) and every finite measure on a Polish space is inner regular (Proposition 1.28-(1)). Therefore $\mathbb{P}^2_1(\_ \mid y)_{y \in X_2}$ is a Radon measure for all $y \in X_2$. ■

Remark 4.28 *Define $w_\sigma(\mathcal{G}) := \min\{card(\mathcal{S}) \mid$ Every set in $\mathcal{G}$ is the countable union of sets in $\mathcal{S}\}$. That is, a sort of countable notion of weight for the topology. If $w_\sigma(\mathcal{G}) \preceq \aleph_1$, then $\mathcal{B}X$ is $\aleph_1$-generated. If X is second countable, obviously $w_\sigma(\mathcal{G}) \preceq \aleph_0$.*

The last theorem can be rephrased in several ways choosing X and X_2 to vary upon topological spaces or continuous dcpo's. The following corollary seems to be the most interesting consequence. Recall from Lemma 2.9 that a continuous dcpo is second countable if and only if it is ω-continuous.

Corollary 4.29 ***Existence of disintegrations and factorizations on***

continuous dcpo's. *For $i = 1, 2$ suppose that $(D_i, \sqsubseteq_i)$ is a continuous dcpo and $\nu_i \in \mathbf{P}^1 D_i$. Let $\overline{\nu}_i$ denote the unique extension of ν_i to a perfect τ-smooth probability measure on $\mathcal{B}_\sigma D_i$. If $\mathbb{P}$ is any joint probability measure on $\mathcal{B}_\sigma D_1 \otimes \mathcal{B}_\sigma D_2$, then*

1. *If $\mathcal{B}_\sigma D_1$ is $\aleph_1$-generated, then there exists a $(\mathcal{B}_\sigma D_2)$-factorization of $\mathbb{P}$.*

Moreover suppose that D_1 is coherent or ω-continuous hence $\overline{\nu}_1$ can be assumed to be a Radon measure on $(D_1, \mathcal{B}_\lambda D_1)$. Let $(D_2, \mathcal{M}(\overline{\nu}_2), \widehat{\nu}_2)$ denote the completion of $(D_2, \mathcal{B}_\sigma D_2, \overline{\nu}_2)$. Then,

2. *Every joint probability measure $\mathbb{P}$ on $\mathcal{B}_\lambda D_1 \otimes \mathcal{M}(\overline{\nu}_2)$ has a $\widehat{\nu}_2$-disintegration $\mathbb{P}_1^2(\_ \mid y)_{y \in D_2}$.*

3. *If D_1 is ω-continuous, then every joint probability measure $\mathbb{P}$ on $\mathcal{B}_\lambda D_1 \otimes \mathcal{B}_\sigma D_2$ has a $\overline{\nu}_2$-disintegration $\mathbb{P}_1^2(\_ \mid y)_{y \in D_2}$.*

In both cases $\mathbb{P}_1^2(\_ \mid y)$ is a Radon measure for all $y \in D_2$. Hence the restriction of $\mathbb{P}_1^2(\_ \mid y)$ to $\mathcal{G}_\sigma D_1$ is a continuous valuation on D_1 for all $y \in D_2$.

Suppose that $(D_i, \sqsubseteq_i)$ and ν_i are as in the previous corollary for $i = 1, 2$. Let ν be a continuous valuation on $(D_1 \times D_2, \mathcal{G}_\sigma(D_1 \times D_2))$ and suppose that $\nu(\pi_i^{-1} G) = \nu_i(G)$ for all $G \in \mathcal{G}_\sigma D_i$. Let $\overline{\nu}$ be the extension of ν to a probability measure on $(D_1 \times D_2, \mathcal{B}_\sigma(D_1 \times D_2))$. Since $\pi_i[\overline{\nu}](G) = \nu_i(G)$ for all $G \in \mathcal{G}_\sigma D_i$, and $\mathcal{G}_\sigma D_i$ is a π-system generating $\mathcal{B}_\sigma D_i$, then (by Proposition 1.23-(3)) the measures $\overline{\nu}_i$ are precisely the marginals of $\overline{\nu}$. Therefore, if D_1 is an ω-continuous or coherent domain, it follows from the last corollary that we can find a function $\nu_1^2(\_ \mid \_) : \mathcal{G}_\sigma D_1 \times D_2 \to [0, 1]$ satisfying:

1. $\nu_1^2(\_ \mid y) : \mathcal{G}_\sigma D_1 \to [0, 1]$ is a normalised continuous valuation for all $y \in D_2$.

2. $\nu_1^2(G \mid \_) : D_2 \to [0, 1]$ is a $\mathcal{M}(\overline{\nu}_2)$-measurable function.

3. For all $G \in \mathcal{G}_\sigma D_1$ and $H \in \mathcal{G}_\sigma D_2$ we have that
$$\nu(G \times H) = \int_H \nu_1^2(G \mid y) \overline{\nu}_2(dy)$$

Now the crucial question becomes to determine in which cases the function $\nu_1^2(G \mid \_) : D_2 \to [0, 1]$ in (2) is Scott continuous, hence the integral in (3) can be replaced by a domain theoretic integral (see Sections 6.4 and 6.5).

Remark 4.30 *Recall from Remark 1.19 that $\mathcal{B}_\sigma D_1 \otimes \mathcal{B}_\sigma D_2 \subseteq \mathcal{B}_\sigma(D_1 \times D_2)$ but in general they are not the same. Hence, the fact that ν is a continuous* *valuation on $(D_1 \times D_2, \mathcal{G}_\sigma(D_1 \times D_2))$ with "marginals" ν_i is a stronger requirement than only having a joint probability measure $\mathbb{P}$ for $(D_1, \mathcal{B}_\sigma D_1, \overline{\nu}_1)$ and $(D_2, \mathcal{B}_\sigma D_2, \overline{\nu}_2)$.*

4.5 Another definition of disintegration

In Section 10.30 of [HJ94] disintegration is defined in a different way. Suppose that (Ω, Γ, ν) is a measure space. The measure ν is **sum-finite** if there exists a sequence (ν_n) of finite measures on (Ω, Γ) such that $\nu = \sum_{n\in\mathbb{N}} \nu_n$. Suppose that $(X_1, \Sigma_1, \mathbb{P})$ and $(X_2, \Sigma_2, \mathbb{P}_2)$ are probability spaces and $T : X_1 \to X_2$ is a measurable function. We say that $\xi(\_ \mid \_) : \Sigma_1 \times X_2 \to \overline{\mathbb{R}}_+$ *disintegrates $\mathbb{P}$ along T with respect to $\mathbb{P}_2$ if:

1. $\xi(\_ \mid y)$ is a sum-finite measure on (X_1, Σ_1) for all $y \in X_2$.
2. $\xi(E \mid \_) : X_2 \to \overline{\mathbb{R}}_+$ is $\mathcal{M}(\mathbb{P}_2)$-measurable for all $E \in \Sigma_1$.
3. For all $E \in \Sigma_1$ and $F \in \Sigma_2$

$$\mathbb{P}(E \cap T^{-1}F) = \textstyle\int_F \xi(E \mid y)\mathbb{P}_2(dy)$$

Remark If there exists a random function $T : X_1 \to X_2$ such that $\mathbb{P}_2 = \mathbb{P}_T$, then $F : X_1 \to X_1 \times X_2$ given by $F(x) := (x, T(x))$ is a random function such that $\mathbb{P}_F$ is a joint distribution for $(X_1, \Sigma_1, \mathbb{P})$ and $(X_2, \Sigma_2, \mathbb{P}_2)$. Conversely if there exists a random function $F : X_1 \to X_1 \times X_2$ such that $\mathbb{P}_F$ is a joint distribution, then $F(x) = (x, T(x))$ for some $T : X_1 \to X_2$ satisfying $\mathbb{P}_2 = \mathbb{P}_T$.

Let Id denote the identity map. Disintegration and *disintegration are related as follows.

Claim: Suppose that $(X_1, \Sigma_1, \mathbb{P})$ and $(X_2, \Sigma_2, \mathbb{P}_2)$ are probability spaces. If there exists a random function $T : X_1 \to X_2$ and a $\mathbb{P}_2$-disintegration $\mathbb{P}_1^2(_ \mid y)_{y\in X_2}$of $\mathbb{P}_{(Id,T)}$ on $(X_1 \times X_2, \Sigma_1 \otimes \mathcal{M}(\mathbb{P}_2))$, then $\mathbb{P}_1^2(\_ \mid \_)$ *disintegrates $\mathbb{P}$ along T with respect to $\mathbb{P}_2$.

Proof Notice that

1. $\mathbb{P}_1^2(\_ \mid y)$ is a probability measure on (X_1, Σ_1) for all $y \in X_2$.
2. $\mathbb{P}_1^2(E \mid \_) : X_2 \to [0, 1]$ is a $\mathcal{M}(\mathbb{P}_2)$-measurable function for all $E \in \Sigma_1$.
3. For all $E \in \Sigma_1$ and $F \in \Sigma_2 \subseteq \mathcal{M}(\mathbb{P}_2)$

$$\begin{aligned}\mathbb{P}(E \cap T^{-1}F) &= \mathbb{P}(Id \in E, T \in F)\\ &= \mathbb{P}_{(Id,T)}(E \times F)\\ &= \int_F \mathbb{P}_1^2(E \mid y)\mathbb{P}_2(dy)\end{aligned}$$

Note that from (3) we have that $\mathbb{P}(T^{-1}F) = \mathbb{P}(X_1 \cap T^{-1}F) = \int_F \mathbb{P}_2(dy) = \mathbb{P}_2(F)$, hence $\mathbb{P}_2 = \mathbb{P}_T$. ■

In terms of regular conditional probability distributions the previous claim can be stated as follows.

Claim: Suppose that $(X_1, \Sigma_1, \mathbb{P})$ and $(X_2, \Sigma_2, \mathbb{P}_2)$ are probability spaces. If there exists a random function $T : X_1 \to X_2$ such that $\mathbb{P}_2 = \mathbb{P}_T$ and an as-rcpd $\mathbb{P}_{Id}^T(\_ \mid \_)$ of Id given T, then $\mathbb{P}_{Id}^T(\_ \mid \_)$ *disintegrates $\mathbb{P}$ along T with respect to $\mathbb{P}_2$.

Proof Notice that $\mathbb{P}_{Id}^T(\_ \mid \_) : \Sigma_1 \times X_2 \to [0,1]$ and

1. $\mathbb{P}_{Id}^T(\_ \mid y)$ is a probability measure on (X_1, Σ_1) for all $y \in X_2$.
2. $\mathbb{P}_{Id}^T(E \mid \_) : X_2 \to [0,1]$ is $\mathcal{M}(\mathbb{P}_T)$-measurable for all $E \in \Sigma_1$.
3. For all $E \in \Sigma_1$ and $F \in \Sigma_2$

$$\begin{aligned}\mathbb{P}(E \cap T^{-1}F) &= \mathbb{P}(Id \in E, T \in F)\\ &= \int_F \mathbb{P}_{Id}^T(E \mid y)\mathbb{P}_T(dy)\end{aligned}$$

Since $\mathbb{P}_2 = \mathbb{P}_T$, the conclusion follows. ■

5. Domain representations

5.1 Summary

The topic of this chapter is to embed a topological space $(X, \mathcal{G})$ as a subspace of a continuous dcpo $(D, \sqsubseteq)$ in such a way that the subspace Scott topology on X agrees with $\mathcal{G}$. In the light of Proposition 2.5, this enables us to approximate a (locally finite) continuous valuation on X with a directed family of simple valuations on D. In later chapters we discuss the impact of such results on domain theoretic integration. We present two constructions. The first one (Theorem 5.18) is a simple application of the existing theory. It embeds any locally compact sober space $(X, \mathcal{G})$ as a subspace of a continuous lattice $(A(X), \sqsubseteq)$ in such way that $\mathcal{G}$, $\mathcal{G}_{cc}$, $\mathcal{G}_p$ agree with the relative Scott, lower and Lawson topologies on $A(X)$.

The second construction (Theorems 5.28 and 5.27) shows that for any locally compact sober space $(X, \mathcal{G})$ there exists a continuous dcpo $(\Lambda X, \sqsubseteq)$ such that the patch space $(X, \mathcal{G}_p)$ is homeomorphic to $\max(\Lambda X)$ with the relative Scott and Lawson topologies of ΛX. Furthermore, if X is stably locally compact, then ΛX is a coherent domain; if X is second countable, then ΛX is ω-continuous; and if X is stably compact, then ΛX has a bottom element. If X is stably locally compact, then $(Lens(X), \supseteq)$ is also a coherent domain with the same properties as ΛX. If X is locally compact and Hausdorff, then $(Lens(X), \supseteq)$ agrees with the upper space of X; that is, the continuous dcpo consisting of the nonempty compact subsets of X ordered by reverse inclusion (Remark 5.33).

Recall that $\mathcal{G}_{cc}X$, $\mathcal{G}_pX$ are the cocompact and patch topologies on X and for $\bullet = cc, p$ we set $X^\bullet = (X, \mathcal{G}_\bullet)$. If $(D, \sqsubseteq)$ is a poset, then $\mathcal{G}_\sigma D$, $\mathcal{G}_l D$, $\mathcal{G}_\lambda D$ are the Scott, lower and Lawson topologies on D with $D^\bullet = (D, \mathcal{G}_\bullet D)$ for $\bullet = \sigma, l, \lambda$. We introduce the following definition.

Definition 5.1 *Let $(X, \mathcal{G})$ be a T_0 topological space and $(D, \sqsubseteq)$ be a continuous dcpo. We say that $(D, \sqsubseteq, e)$ is a **Scott domain representation** (S-domain representation) for X if X embeds as a subspace of D^σ and $e \in [X \to D^\sigma]$ is the embedding function. We set $\widetilde{x} := e(x)$ for all $x \in X$ and $\widetilde{B} = e[B]$ for all $B \subseteq X$. If D has a least element $\perp$, we say that*

$(D, \sqsubseteq, e, \bot)$ is a ***pointed*** *S-domain representation. If $e[X]$ is dense in D, then D is a* ***dense*** *domain representation for X.*

We call these structures S-domain representations to distinguish them from other domain representations for topological spaces [SHT95][SSH97][Bla97b]-[Bla97a]. If $(D, \sqsubseteq, e)$ is an S-domain representation for X and $\max(D) = e[X]$, Martin [Mar98][Mar00] calls D a **model** for X.

5.2 Auxiliary results

Recall from Proposition 1.10 that for a continuous dcpo $(D, \sqsubseteq)$ topologised with $\mathcal{G}_\sigma D$ we have that $\mathcal{G}_l D = \mathcal{G}_{cc} D$ and $\mathcal{G}_\lambda D = \mathcal{G}_p D$. Thus, D is a stably compact domain if and only if it is compact Hausdorff in the Lawson topology. In particular, this holds for continuous lattices [GHK$^+$80, Theorem III-1.10] and for coherent domains with least element. We make use of the following definitions and auxiliary results. A **compactification** for a topological space $(X, \mathcal{G})$ is a subspace embedding $\xi : X \to Y$ where $(Y, \mathcal{H})$ is a compact space and $\xi[X]$ is a dense subset of Y. A poset $(P, \sqsubseteq)$ is called a (meet) **semilattice** if every finite nonempty subset of P has a greatest lower bound. A semilattice which is a continuous dcpo is called a **continuous semilattice.** Let $(L, \sqsubseteq, \wedge)$ be a semilattice. We say that $x \in L$ is **prime** if $a \wedge b \sqsubseteq x$ implies $a \sqsubseteq x$ or $b \sqsubseteq x$ for all $a, b \in L$. We define the **spectrum** *Spec* L of L as the set of all non-maximum primes of L.

Proposition 5.2 *Suppose that $(D, \sqsubseteq)$ is a dcpo, I is a directed set and $\langle x(i,j)\rangle_{(i,j)\in I\times I}$ is a monotone net in D. Then*

$$\textstyle\bigsqcup^{\uparrow}_{(i,j)\in I\times I} x_{(i,j)} = \bigsqcup^{\uparrow}_{i\in I} \bigsqcup^{\uparrow}_{j\in I} x_{(i,j)} = \bigsqcup^{\uparrow}_{j\in I} \bigsqcup^{\uparrow}_{i\in I} x_{(i,j)} = \bigsqcup^{\uparrow}_{i\in I} x_{(i,i)}$$

Proof See Proposition 2.1.12 in [AJ94]. ■

Proposition 5.3 *Suppose that $(D_1, \sqsubseteq_1)$ and $(D_2, \sqsubseteq_2)$ are continuous (ω-continuous) dcpo's. Then*

- *$D_1 \times D_2$ with the order $(x_1, y_1) \sqsubseteq (x_2, y_2)$ if and only if $x_1 \sqsubseteq_1 x_2$ and $y_1 \sqsubseteq_2 y_2$ is a continuous (ω-continuous) dcpo.*
- *$(x_1, y_1) \ll (x_2, y_2)$ if and only if $x_1 \ll_1 x_2$ and $y_1 \ll_2 y_2$.*
- *If B_i is a basis for D_i, then $B_1 \times B_2$ is a basis for $D_1 \times D_2$.*

Proof See Proposition 3.2.4 in [AJ94]. ■

Recall that a dcpo $(D, \sqsubseteq)$ is bounded complete if every finite nonempty bounded subset has a least upper bound.

Proposition 5.4 *Suppose that $(D, \sqsubseteq)$ is a continuous dcpo. Then,*

1. *D is coherent if and only if for all A, B finite subsets of D we have that $(\uparrow A \cap \uparrow B) \in \mathcal{Q}_\sigma D$.*

2. *D is coherent if and only if for all $a, b \in D$ we have that $(\uparrow a \cap \uparrow b) \in \mathcal{Q}_\sigma D$.*

3. *If D is bounded complete, then D is a coherent domain.*

Proof (1) See part (2) of Proposition 4.38 in [Mis98]. For (2) notice that if $A = \{a_1,..., a_n\}$ and $B = \{b_1,..., b_m\}$, then

$$\begin{aligned} \uparrow A \cap \uparrow B &= (\textstyle\bigcup_i \uparrow a_i) \cap (\bigcup_j \uparrow b_j) \\ &= \textstyle\bigcup_{i,j} (\uparrow a_i \cap \uparrow b_j). \end{aligned}$$

The claim follows from part (1). For (3) suppose that $a, b \in D$. Observe that if $\uparrow a \cap \uparrow b \neq \emptyset$, then $\{a, b\}$ is bounded and it follows by assumption that $a \vee b$ exists. Clearly $\uparrow a \cap \uparrow b = \uparrow(a \vee b)$. Hence $(\uparrow a \cap \uparrow b) \in \mathcal{Q}_\sigma D$ and the conclusion is immediate from part (2). ■

5.3 A review of some existing S-domain representations

We survey some of the existing S-domain representations to give the reader some perspective for the constructions defined in the next sections. This survey does not intend to be exhaustive.

5.3.1 The Sierpinski Lattice

The **Sierpinski space** is defined as the lattice $(2 = \{0, 1\}, \leq)$ with the Scott topology. That is $\mathcal{G}_\sigma = \{\emptyset, \{1\}, \{0, 1\}\}$.

Definition 5.5 *Suppose that $(X, \mathcal{G})$ is a T_0 topological space. The* ***Sierpinski lattice*** *associated with X is defined as $\mathbb{S}X = \Pi_{i \in \mathcal{G}} 2_i$ where $2_i = (2, \mathcal{G}_\sigma)$ for all $i \in \mathcal{G}$. Hence the product topology $\otimes_{i \in \mathcal{G}} (\mathcal{G}_\sigma 2_i)$ on $\mathbb{S}X$ is the Scott topology $\mathcal{G}_\sigma \mathbb{S}$ corresponding to the pointwise order: $(x_i)_{i \in \mathcal{G}} \sqsubseteq (y_i)_{i \in \mathcal{G}}$ if $x_i \leq y_i$ for all $i \in \mathcal{G}$.*

Let the function $e : X \to \mathbb{S}X$ be defined as $e(x) = (x_i)_{i \in \mathcal{G}}$ where

$$x_i = \begin{cases} 1 & \text{if } x \in i \\ 0 & \text{otherwise} \end{cases}$$

That is, $e(x) = (\delta_x(i))_{i \in \mathcal{G}} = (\mathbb{I}_i(x))_{i \in \mathcal{G}}$.

Proposition 5.6 *Suppose that $(X, \mathcal{G})$ is a T_0 topological space. Then e :*

$X \to (\mathbb{S}X)^\sigma$ is a subspace embedding. Hence $(\mathbb{S}X, \mathcal{G}_\sigma\mathbb{S}, e)$ is an S-domain representation for X.

Proof See Lemma II-3.4 in [GHK⁺80]. ■

Several remarks are in order.

(1) $(\mathbb{S}X, \mathcal{G}_\sigma\mathbb{S})$ is not only a continuous lattice but furthermore an algebraic lattice. In particular, it is a stably compact space.

(2) For $j \in \mathcal{G}$, consider the cylinder $\langle\{1\}_j\rangle$. That is,

$$\langle\{1\}_j\rangle = \{(x_i)_{i\in\mathcal{G}} \in \mathbb{S}X \mid x_j = 1\}.$$

Then $e^{-1}\langle\{1\}_j\rangle = j$. In the same way $e^{-1}\langle\{1\}_j, \{1\}_k\rangle = j \cap k = e^{-1}\langle\{1\}_{j\cap k}\rangle$. This means that, in general, an open set in $\mathcal{G}$ is the inverse image of many open sets in $\mathbb{S}X$. In particular if $j \cap k = \emptyset$, then $e^{-1}\langle\{1\}_j, \{1\}_k\rangle = \emptyset$ but $\langle\{1\}_j, \{1\}_k\rangle$ need not be empty. This shows that $e[X]$ is in general not dense in $\mathbb{S}X$.

(3) For every $j \in \mathcal{G}$ let the element $j^\natural \in \mathbb{S}X$ be defined as $j^\natural = (x_i)_{i\in\mathcal{G}}$ where

$$x_i = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

We claim that $j^\natural \in e[X]$ if and only if X has a bottom element $\perp$, $j^\natural = e(\perp)$ and $j = X$. This can be seen as follows: suppose that $j^\natural = e(x)$ for some $x \in X$. Then $x \in j$ but $x \notin k$ for all k with $j \subsetneqq k$. This means that j is not properly contained in any open set, hence $j = X$. Similarly, $x \notin k$ for all $k \subsetneqq j = X$. Now notice that for every $y \in X$, the open set $(cl\{y\})^\mathsf{c}$ is properly contained in X. Hence for all $y \in X$ we have that $x \notin (cl\{y\})^\mathsf{c}$; that is, $x \in {\downarrow}y$. Therefore $x = \perp$.

(4) Suppose that ν is a finite continuous valuation on X, then $\nu_e = \nu \circ e^{-1}$ is the least upper bound of the directed family of simple valuations $\{\sum_{j\in B} \nu(j)\delta_{j^\natural} \mid B \subseteq_f \mathcal{G}\}$. Nevertheless $\delta_{j^\natural}$ is rarely equal to ν_e for a continuous valuation ν on X.

(5) Let $cl_\sigma(\widetilde{X})$ denote the closure of $\widetilde{X} = e[X]$ as a subset of $(\mathbb{S}X, \mathcal{G}_\sigma\mathbb{S})$. Then $cl_\sigma(\widetilde{X})$ endowed with the subspace topology from $\mathbb{S}X$ is a compactification for X.

(6) Recall that $\mathbb{S}X$ topologised with the Lawson topology $\mathcal{G}_\lambda\mathbb{S}$ is a compact Hausdorff space. So we could also consider the closure $cl_\lambda(\widetilde{X})$ under $\mathcal{G}_\lambda\mathbb{S}$ (the Lawson topology on $\mathbb{S}X$) endowed with the subspace topology from

$(\mathbb{S}X, \mathcal{G}_\sigma\mathbb{S})$ (yes, the Scott topology). This construction is called by Lawson [Law91, Example 31] a strongly sober compactification and by Kopperman [Kop95, Definition 6.5] a skew compactification for X.

5.3.2 The upper space

The idea of using compact subsets as a computational model for Hausdorff spaces was originally introduced by A. Edalat in [Eda95d]; we survey the results in *loc cit* which relate to domain representations. Let $(X, \mathcal{G})$ be a topological space. For every $G \in \mathcal{G}$ define the following sets:

$$\begin{aligned} \Box G &= \{E \subseteq X \mid E \subseteq G\} \\ \Diamond G &= \{E \subseteq X \mid E \cap G \neq \emptyset\} \end{aligned}$$

The **upper Vietoris topology** $\mathcal{G}_{uV}$ and the **lower Vietoris topology** $\mathcal{G}_{lV}$ in 2^X are defined as $\mathcal{G}_{uV} := \mathcal{T}(\{\Box G \mid G \in \mathcal{G}\})$ and $\mathcal{G}_{lV} := \mathcal{T}(\{\Diamond G \mid G \in \mathcal{G}\})$. The **Vietoris** topology on X is the join $\mathcal{G}_V := \mathcal{G}_{uV} \vee \mathcal{G}_{lV}$ and $(2^X, \mathcal{G}_V)$ is called the **Vietoris hyperspace** of X (see Section 7.4 in [Smy92b], for example).

Definition 5.7 *Let $(X, \mathcal{G})$ be a Hausdorff space. The* ***upper space*** $\mathbf{U}X$ *of X is defined as $\mathcal{K}X\backslash\{\emptyset\}$ with the subspace topology $\mathcal{G}_{uV}(\mathbf{U}X)$ of the upper Vietoris hyperspace $(2^X, \mathcal{G}_{uV})$. That is, $\mathcal{G}_{uV}(\mathbf{U}X)$ is the topology generated by the sets $\Box G = \{K \in \mathcal{K} \mid G \supseteq K\}$ with $G \in \mathcal{G}$.*

Similarly we define $\mathcal{G}_{lV}(\mathbf{U}X)$ to be the topology generated by the sets $\Diamond G = \{K \in \mathcal{K} \mid K \cap G \neq \emptyset\}$ with $G \in \mathcal{G}$ and $\mathcal{G}_V(\mathbf{U}X) := \mathcal{G}_{uV}(\mathbf{U}X) \vee \mathcal{G}_{lV}(\mathbf{U}X)$. Let the function $e : X \to \mathbf{U}X$ be defined as $e(x) := \{x\}$ and $\mathbf{U}_\perp X = \mathbf{U}X \cup \{X\}$. The following result was shown by A. Edalat although part of the construction follows from Proposition 5.12 below and Exercise 7.7.8-15 in [Smy92b].

Proposition 5.8 *Suppose that $(X, \mathcal{G})$ is a Hausdorff space. Then,*

1. *$(\mathbf{U}X, \mathcal{G}_{uV}(\mathbf{U}X))$ is an mc-space and the specialisation order $\sqsubseteq_u$ is given by $K_1 \sqsubseteq_u K_2 \Leftrightarrow K_1 \supseteq K_2$.*
2. *$(\mathbf{U}X, \supseteq)$ is a bounded complete dcpo.*
3. *$e : X \to \mathbf{U}X$ is a dense subspace embedding where $e[X] = \max(\mathbf{U}X)$.*
4. *If X is locally compact, then*
    - *$(\mathbf{U}X, \supseteq)$ is a coherent domain where $K_1 \ll K_2$ if and only if $int(K_1) \supseteq K_2$.*

- $(\mathbf{U}_\perp X, \supseteq)$ *is compact in the Lawson topology* $\mathcal{G}_\lambda(\mathbf{U}_\perp X)$. *If* X *is compact, then obviously* $\mathbf{U}_\perp X = \mathbf{U}X$.
- *If* X *is second countable, then* $(\mathbf{U}X, \supseteq)$ *is* ω*-continuous.*
- *The upper Vietoris topology* $\mathcal{G}_{uV}(\mathbf{U}X)$ *on* $\mathbf{U}X$ *agrees with the Scott topology.*
- $\mathcal{G}_\lambda(\mathbf{U}X) \restriction_{\max(\mathbf{U}X)} = \mathcal{G}_\sigma(\mathbf{U}X) \restriction_{\max(\mathbf{U}X)}$. *That is, the relative Scott and Lawson topologies agree on the maximal elements.*

Proof See Propositions 3.1, 3.3 and 3.4 in [Eda95d]. ■

Remark 5.9 *Suppose that* $(X, \mathcal{G})$ *is a compact Hausdorff space. Then* ($\mathcal{F} = \mathcal{K}$ *and) the Fell topology* $\mathcal{G}_F(\mathcal{F})$, *the Vietoris topology* $\mathcal{G}_V(\mathbf{U}X)$ *and the Lawson topology* $\mathcal{G}_\lambda(\mathbf{U}X)$ *agree.*

Proof If X is compact Hausdorff then, $\mathcal{F} = \mathcal{K}$ and

$$\begin{aligned} \{F \in \mathcal{F} \mid F \cap Q = \emptyset\}_{Q \in \mathcal{K}} &= \{K \in \mathcal{K} \mid Q^{\mathsf{c}} \supseteq K\}_{Q \in \mathcal{K}} \\ &= \{K \in \mathcal{K} \mid G \supseteq K\}_{G \in \mathcal{G}} \end{aligned}$$

and

$$\{F \in \mathcal{F} \mid F \cap G \neq \emptyset\}_{G \in \mathcal{G}} = \{K \in \mathcal{K} \mid F \cap G \neq \emptyset\}_{G \in \mathcal{G}}.$$

Hence, the Fell topology $\mathcal{G}_F(\mathcal{F})$ and the Vietoris topology $\mathcal{G}_V(\mathbf{U}X)$ agree. By Proposition 3.14 we know that the Lawson topology on $(\mathcal{F}, \supseteq)$ agrees with $\mathcal{G}_F(\mathcal{F})$. But $\mathcal{F} = \mathcal{K} = \mathbf{U}X$. ■

Regarding the agreement of the Fell and Vietoris topologies much more can be said; the reader is referred to [NS96] for an in-depth treatment.

Remark 5.10 *Suppose that* $(X, \mathcal{G})$ *is a locally compact Hausdorff space. Then* $\widetilde{X} = \max(\mathbf{U}X)$ *is closed in* $(\mathbf{U}X)^\lambda$.

Proof Let $\overline{X}$ denote the closure of $\widetilde{X}$ in $(\mathbf{U}X)^\lambda$. Take any $K \in \mathcal{K}X$ with $K \neq \widetilde{y}$ for all $y \in X$. By local compactness there exists $K_2 \in \mathcal{K}X$ with $K_2 \ll K$. Choose any $x \in K_2$. Since K contains at least two elements u_1, u_2, we can find an open set $G_x \in \mathcal{G}$ such that $x \in G_x$ but $u_i \notin G_x$ for one $i \in \{1, 2\}$ (notice that x could be the other u_i). By local compactness there exists $Q_x \in \mathcal{K}X$ such that $x \in int(Q_x) \subseteq Q_x \subseteq G_x$. Then $\{int(Q_x)\}_{x \in K_2}$ is an open covering for the compact set K_2 such that $K \nsubseteq Q_x$ for all x. Extract a finite subcovering $\{int(Q_j)\}_{j \in J}$ and let $F := \bigcup_{j \in J} \uparrow_{\mathbf{U}} Q_j$. Then F is a (lower) closed set in $(\mathbf{U}_\perp X)^\lambda$ and $K \notin F$. Since $K_2 \subseteq \bigcup_{j \in J} Q_j$, then for every $x \in K_2$, there exists $j \in J$ such that $Q_j \supseteq \{x\}$; hence $\widetilde{x} \in F$.

Therefore $\Uparrow_{\mathbf{U}} K_2 \cap F^{\mathsf{c}}$ is an open set in $(\mathbf{U}_\perp X)^\lambda$ which does not contain any elements of $\widetilde{X}$. Since $K \in (\Uparrow_{\mathbf{U}} K_2 \cap F^{\mathsf{c}})$ it follows that $K \notin \overline{X}$. ■

The reader should notice that the previous remark is stated for $\mathbf{U}X$ and not for $\mathbf{U}_\perp X$. The next remark puts things into perspective.

Remark 5.11 *Suppose that $(X, \mathcal{G})$ is a locally compact Hausdorff space which is not compact. Then the closure of $\widetilde{X} = e[X]$ in $(\mathbf{U}_\perp X)^\lambda = (\mathbf{U}_\perp X, \mathcal{G}_\lambda(\mathbf{U}_\perp X))$ is the Alexandroff one point compactification of X.*

Proof Let $\overline{X}$ denote the closure of $\widetilde{X}$ in $(\mathbf{U}_\perp X)^\lambda$. Since $(\mathbf{U}_\perp X)^\lambda$ is compact Hausdorff, then $\overline{X}$ is a compact subset of $(\mathbf{U}_\perp X)^\lambda$. By the previous remark we know that if $K \in \mathcal{K}X$ then $K \notin \overline{X}$, hence $\overline{X} \subseteq \widetilde{X} \cup \{\perp\}$. We claim that $\perp \in \overline{X}$. Notice that $\uparrow\perp = \mathbf{U}_\perp X$. Hence the open neighborhoods of $\perp$ are either $\mathbf{U}_\perp X$ or sets of the form $(\bigcup_{j=1}^n \uparrow_{\mathbf{U}} K_j)^{\mathsf{c}}$ with $K_j \in \mathcal{K}X$ for $j = 1,...,n$. But $X \nsubseteq \bigcup_{j=1}^n K_j$, since X is not compact. Therefore $\widetilde{X} \nsubseteq \bigcup_{j=1}^n \uparrow_{\mathbf{U}} K_j$. Hence there exists $\widetilde{x} \in (\bigcup_{j=1}^n \uparrow K_j)^{\mathsf{c}}$. It follows that $\perp \in \overline{X}$. Therefore $\overline{X} = \widetilde{X} \cup \{\perp\}$ is a compact Hausdorff space. Since the one point compactification is essentially unique and $\widetilde{X}$ is homeomorphic to X the conclusion follows. ■

5.3.3 The spectrum of $\mathcal{Q}$

The following construction can be seen as a generalisation of the upper space for non-Hausdorff spaces. For the relationship with the Smyth powerdomain see Section 6.2.3 in [AJ94]. For a topological space $(X, \mathcal{G})$, let $\mathcal{Q}$ be given the upper Vietoris topology. Let the function $e : X \to \mathcal{Q}$ be defined as $e(x) := sat(\{x\}) = \uparrow x$.

Proposition 5.12 *Suppose that $(X, \mathcal{G})$ is a sober space. Then*

1. *$(\mathcal{Q}\backslash\{\emptyset\}, \supseteq)$ is a dcpo.*
2. *If X is coherent, then $(\mathcal{Q}\backslash\{\emptyset\}, \supseteq)$ is bounded complete.*
3. *The map $e : X \to \mathcal{Q}$ is a dense subspace embedding.*
4. *If X is locally compact, then*
   - *$(\mathcal{Q}, \supseteq, \cup)$ is a continuous semilattice with maximum element $\emptyset$. Hence $(\mathcal{Q}\backslash\{\emptyset\}, \supseteq)$ is a continuous dcpo.*
   - *$e[X] = Spec\ \mathcal{Q}$.*
   - *The Scott topology agrees with the upper Vietoris topology on $\mathcal{Q}$.*

Proof (1) was stated in part (4) of Proposition 3.9 (see also exercise 7.7.8-15 in [Smy92b]). (2) is clear from part (1) of Lemma 3.12. (3) is clear since $\Box G \cap e[X] = e[G]$. The map e is injective since X is T_0. (4) For the first two claims see Corollary 2.17, Lemma 2.23, Proposition 2.24 and Theorem 2.28 in [HM81]. In order to establish the third claim notice that for every $Q \in \mathcal{Q}$, the set $\Uparrow Q = \{Q' \in \mathcal{Q} \mid int(Q) \supseteq Q'\} = \Box int(Q)$ is an upper Vietoris open set. Since $\{\Uparrow Q \mid Q \in \mathcal{Q}\}$ is a base for the Scott topology, it follows that every Scott open is upper Vietoris open. For the other direction, since X is locally compact, $G = \bigcup\{int(Q) \mid G \supseteq Q\}$ for all $G \in \mathcal{G}$. Thus, the set $\Box G$ can be expressed as

$$\begin{aligned}\Box G &= \bigcup\{\Box int(Q) \mid G \supseteq Q\}\\ &= \bigcup_{G \supseteq Q} \Uparrow Q\end{aligned}$$

where $\Uparrow Q$ is a Scott open subset of $(\mathcal{Q}\backslash\{\emptyset\}, \supseteq)$. ■

Hence $(\mathcal{Q}\backslash\{\emptyset\}, \supseteq, sat)$ is an S-domain representation for a locally compact sober space $(X, \mathcal{G})$. For every open set H in $(\mathcal{Q}\backslash\{\emptyset\}, \supseteq)$ define $\nabla H := \bigcup\{int(Q) \mid Q \in H\} \in \mathcal{G}$. It is not difficult to see that $e[\nabla H] = H \cap e[X]$, hence $\nabla H = e^{-1}H$. Note that $G = \nabla \Box G$, while $H \subseteq \Box \nabla H$. In other words, $(\Box, \nabla)$ is an embedding projection pair between $(\mathcal{G}, \supseteq)$ and $(\mathcal{G}_\sigma(\mathcal{Q}), \supseteq)$ and $(\nabla, \Box)$ is an adjunction between $(\mathcal{G}_\sigma(\mathcal{Q}), \subseteq)$ and $(\mathcal{G}, \subseteq)$ (see Definition 3.1.15 and Proposition 3.1.10 in [AJ94]).

5.3.4 Maximal point hulls (MP-hulls)

We summarise some of the results by J. Lawson in [Law97] (see also [Law98]). Recall that a topological space $(X, \mathcal{G})$ is Polish if it is metrizable with a complete and separable metric.

Definition 5.13 *Suppose that (X, d) is a separable metric space. An ω-continuous dcpo $(D, \sqsubseteq)$ is called a maximal point hull (**MP-hull**) for X, if X is homeomorphic to $\max(D)$ as a subspace of D with respect to the Lawson and Scott topologies. In other words, there exists a subspace embedding $e : X \to D^\lambda$ such that $e[X] = \max(D)$ and $\mathcal{G}_\lambda D \restriction_{\max(D)} = \mathcal{G}_\sigma D \restriction_{\max(D)}$.*

Notice that if $e[X] = \max(D)$, then $e[X]$ is dense in D^σ.

Proposition 5.14 *(Lawson)Suppose that (X, d) is a separable metric space. Then,*

1. *X is a Polish space if and only if there exists a MP-hull for X.*
2. *If $(D, \sqsubseteq)$ is an ω-continuous dcpo and $\mathcal{G}_\lambda D \restriction_{\max(D)} = \mathcal{G}_\sigma D \restriction_{\max(D)}$,*

then

- $\max(D)$ *with the subspace topology is a Polish space and a* G_δ *subset of* D^σ *(and hence of* D^λ*).*
- *There exists a compact metrizable pospace* $(Y, \mathcal{H}, \leq)$ *and an order preserving subspace embedding* $j \in [D^\lambda \to Y]$ *such that* $j[\max(D)] \subseteq \max(Y)$.

3. *If* $(D, \sqsubseteq)$ *is an* ω*-continuous dcpo and* $(D, \mathcal{G}_\lambda D)$ *is compact, then* $\mathcal{G}_\lambda D \restriction_{\max(D)} = \mathcal{G}_\sigma D \restriction_{\max(D)}$.

Proof See Theorems 3.7, 3.3 and Corollaries 3.4, 3.5, 3.6 in [Law97]. ■

Notice that part (2) of the previous proposition together with part (4) of Proposition 5.8 imply that every locally compact second countable Hausdorff space is Polish (Corollary 1.27).

5.3.5 Spaces of formal balls

Formal balls were originally introduced by K. Weihrauch and U. Schreiber in [WS81]. Our exposition and results are extracted from the work of Edalat and Heckmann [EH98]. Spaces of formal balls can be seen as an extension of MP-hulls to general metric spaces.

Definition 5.15 *Let* (X, d) *be a metric space. The poset* $(\mathbf{B}X, \sqsubseteq)$ *of* ***formal balls*** *of* X *is defined as* $\mathbf{B}X = X \times \mathbb{R}^+$ *with* $(x, r) \sqsubseteq (y, s)$ *if and only if* $d(x, y) \leq r - s$.

Let $C(x, r) = \{y \in x \mid d(x, y) \leq r\}$ denote the **closed ball** centered at x with radius r. Then $(x, r) \sqsubseteq (y, s)$ implies that $C(x, r) \supseteq C(y, s)$. As the next example shows, the converse implication does not hold in general, even when X is complete. Consider $\mathbb{N}_+$ with the distance $d(m, n) := 1 + \frac{1}{n+m}$ for all $m \neq n$. Then $(\mathbb{N}_+, d)$ is a complete metric space since every Cauchy sequence is eventually constant. Note that $C(1, 1 + \frac{1}{2}) = \mathbb{N}_+ \supseteq \mathbb{N}_+ \setminus \{1\} = C(2, 1 + \frac{1}{4})$ nevertheless $d(1, 2) = 1 + \frac{1}{3} > \frac{1}{4}$. If $(Y, \|\_\|)$ is a normed vector space over $\mathbb{R}$ (e. g., a Banach space), then $(x, r) \sqsubseteq (y, s) \Leftrightarrow C(x, r) \supseteq C(y, s) \Leftrightarrow \|x - y\| \leq r - s$.

Let the function $e : X \to \mathbf{B}X$ be defined as $e(x) := (x, 0)$.

Proposition 5.16 *Let* (X, d) *be a complete metric space. Then,*

1. $(\mathbf{B}X, \sqsubseteq)$ *is a continuous dcpo where* $(x, r) \ll (y, s)$ *if and only if* $d(x, y) < r - s$.

2. $e : X \to (\mathbf{B}X)^\lambda$ *is a subspace embedding,* $e[X] = \max(\mathbf{B}X)$ *is a* G_δ *subset of* $\mathbf{B}X$*, and* $\mathcal{G}_\lambda(\mathbf{B}X) \restriction_{\max(\mathbf{B}X)} = \mathcal{G}_\sigma(\mathbf{B}X) \restriction_{\max(\mathbf{B}X)}$.

3. $\mathbf{B}X$ *is* ω*-continuous if and only if* X *is separable (hence a Polish space).*

Proof See Theorem 6, Proposition 7, Corollary 10 and Theorem 13 in [EH98]. ■

So, spaces of formal balls give dense S-domain representations for complete metric spaces. The results in [EH98] state other interesting facts worth mentioning: (1) $(\mathbf{B}X, \sqsubseteq)$ is always a continuous poset. (2) $(\mathbf{B}X, \sqsubseteq)$ is a dcpo if and only if (X, d) is complete. (3) If (X, d) is not complete, let $(\overline{X}, \overline{d})$ denote its (metric) completion. Then $(\mathbf{B}\overline{X}, \sqsubseteq)$ is isomorphic to the rounded ideal completion (see [AJ94, Section 2.2.6] for example) of $(\mathbf{B}X, \sqsubseteq)$.

5.4 An S-representation using compact ordered spaces

Consider the following result.

Proposition 5.17 *Let* $(X, \mathcal{G})$ *be a locally compact sober space. Then,*

1. $(\mathcal{G}, \subseteq)$ *is a continuous lattice (moreover a continuous Heyting algebra or continuous frame).*

2. *The map* $\Phi_X : X \to \mathcal{G}$ *given by* $\Phi_X(x) = X \backslash cl(x) = X \backslash {\downarrow} x$ *is an embedding of* X *onto Spec* $\mathcal{G}$ *with the relative lower topology induced from (the continuous lattice)* $(\mathcal{G}, \subseteq)$.

3. *Let* $S =$ *Spec* $\mathcal{G}$*, then*

   - $(X, \mathcal{G}) \cong (S, \mathcal{G}_l(\mathcal{G}) \restriction_S)$
   - $(X, \mathcal{G}_{cc}) \cong (S, \mathcal{G}_\sigma(\mathcal{G}) \restriction_S)$
   - $(X, \mathcal{G}_p) \cong (S, \mathcal{G}_\lambda(\mathcal{G}) \restriction_S)$

Proof See [HL78] and Theorem 4.4 and Proposition 4.5 in [HM81]. ■

In other words $\Phi_X : (X, \mathcal{G}, \mathcal{G}_{cc}) \to (\mathcal{G}, \mathcal{G}_l(\mathcal{G}), \mathcal{G}_\sigma(\mathcal{G}))$ is an embedding of bitopological spaces. The embedding Φ_X reverses the order since $x \sqsubseteq y$ implies $\Phi_X(y) \subseteq \Phi_X(x)$. Therefore the corresponding topologies are in the inverse relation to what we want; that is, we are looking for a continuous dcpo where the Scott topology restricts to the original topology of the space. The fact that $(\mathcal{G}, \subseteq)$ is a continuous lattice implies that $(\mathcal{G}, \mathcal{G}_\lambda(\mathcal{G}))$ is a compact Hausdorff space. If we take the closure of $\Phi_X[X]$ with respect to $\mathcal{G}_\lambda(\mathcal{G})$

we obtain a compact ordered space, which is called the **Fell compactification** of X (see [Law91, Section 7.8]). For simplicity we will make no distinction between X and $\Phi_X[X]$ in the following. Furthermore, we define the **antispectrum** $(a(X), \sqsubseteq)$ of a locally compact sober space $(X, \mathcal{G})$ to be the continuous lattice $(\mathcal{G}, \subseteq)$. Then the last proposition can be rewritten as:

$$\begin{aligned}
&X \subseteq a(X) && \text{(via } \Phi_X\text{)}\\
&\mathcal{G}X = \mathcal{G}_l(a(X)) \restriction_X\\
&\mathcal{G}_{cc}X = \mathcal{G}_\sigma(a(X)) \restriction_X\\
&\mathcal{G}_pX = \mathcal{G}_\lambda(a(X)) \restriction_X
\end{aligned}$$

Now consider the space $(a(X), \mathcal{G}_\sigma(a(X)))$, i.e. the antispectrum seen as a topological space with the Scott topology corresponding to the order $\subseteq$. Then $(a(X), \mathcal{G}_\sigma(a(X)))$ is in particular, a locally compact sober space; so we can consider the antispectrum of this space to obtain

$$\begin{aligned}
&a(X) \subseteq a(a(X))\\
&\mathcal{G}_\sigma(a(X)) = \mathcal{G}_l(a(a(X))) \restriction_{a(X)}\\
&\mathcal{G}_{cc}(a(X)) = \mathcal{G}_\sigma(a(a(X))) \restriction_{a(X)}\\
&\mathcal{G}_p(a(X)) = \mathcal{G}_\lambda(a(a(X))) \restriction_{a(X)}
\end{aligned}$$

Since $(a(X), \sqsubseteq)$ is a continuous lattice, it is in particular a continuous dcpo. Then $\mathcal{G}_{cc}(a(X)) = \mathcal{G}_l(a(X))$ and $\mathcal{G}_p(a(X)) = \mathcal{G}_\lambda(a(X))$. Therefore

$$\begin{aligned}
\mathcal{G}_\sigma(a(a(X))) \restriction_X &= \mathcal{G}_\sigma(a(a(X))) \restriction_{a(X)} \restriction_X\\
&= \mathcal{G}_{cc}(a(X)) \restriction_X\\
&= \mathcal{G}_l(a(X)) \restriction_X\\
&= \mathcal{G}X
\end{aligned}$$

By a similar argument we can verify that

$$\begin{aligned}
&\mathcal{G}_l(a(a(X))) \restriction_X = \mathcal{G}_{cc}X \qquad \text{and}\\
&\mathcal{G}_\lambda(a(a(X))) \restriction_X = \mathcal{G}_pX.
\end{aligned}$$

Then $a(a(X))$ is the continuous dcpo we are looking for. We abbreviate $a(a(X))$ as $A(X)$. The situation is not so complicated as it might seem at first sight. Unwinding the definitions we obtain that $A(X)$ is the Scott topology $\mathcal{G}_\sigma(\mathcal{G})$ of the lattice $(\mathcal{G}, \subseteq)$. If we define e to be the composition of the embeddings $\Phi_X : X \to \mathcal{G}$ and $\Phi_\mathcal{G} : \mathcal{G} \to \mathcal{G}_\sigma(\mathcal{G})$ we obtain

$$\begin{aligned}
e(y) &= \Phi_{\mathcal{G}} \circ \Phi_X(y) \\
&= \Phi_{\mathcal{G}}(X \backslash cl(y)) \\
&= \mathcal{G} \backslash \downarrow_{\mathcal{G}}(X \backslash cl(y)) \\
&= \mathcal{G} \backslash \{H \in \mathcal{G} \mid H \subseteq X \backslash cl(y)\} \\
&= \{H \in \mathcal{G} \mid H \cap cl(y) \neq \emptyset\} \\
&= \{H \in \mathcal{G} \mid y \in H\} \\
&= \eta_X(y)
\end{aligned}$$

Where η_X is the function $y \mapsto \mathcal{N}(y)$ that maps every point in X to its completely prime filter of open neighborhoods (see Section 2.7). Putting everything together we have,

Theorem 5.18 *Let $(X, \mathcal{G})$ be a locally compact sober space. Then*

1. $(A(X), \sqsubseteq) = (\mathcal{G}_\sigma(\mathcal{G}), \subseteq)$ *is a continuous lattice, where* $\mathcal{G}_\sigma(\mathcal{G})$ *is the Scott topology in* $(\mathcal{G}, \subseteq)$.
2. $A(X)$ *with the Lawson topology* $\mathcal{G}_\lambda A(X)$ *is a compact Hausdorff space.*
3. *The map* $\eta_X : X \to A(X)$ *given by* $\eta_X(y) = \{G \in \mathcal{G} \mid y \in G\}$ *is an embedding. If we let* $S = \eta[X]$, *then*
   - $(X, \mathcal{G}) \cong (S, \mathcal{G}_\sigma A(X) \restriction_S)$
   - $(X, \mathcal{G}_{cc}) \cong (S, \mathcal{G}_l A(X) \restriction_S)$
   - $(X, \mathcal{G}_p) \cong (S, \mathcal{G}_\lambda A(X) \restriction_S)$

In other words $\eta_X : (X, \mathcal{G}, \mathcal{G}_{cc}) \to (A(X), \mathcal{G}_\sigma A(X), \mathcal{G}_l A(X))$ is a bitopological embedding.

Corollary 5.19 *Let $(X, \mathcal{G})$ be a locally compact sober space. Then $(A(X), \sqsubseteq, \eta_X)$ is an S-domain representation for $(X, \mathcal{G})$.*

Again for simplicity we write $X \subseteq a(X) \subseteq A(X)$. Since $(a(X), \mathcal{G}_\lambda a(X))$ and $(A(X), \mathcal{G}_\lambda A(X))$ are compact Hausdorff spaces, it follows that $a(X)$ is a compact subset of $(A(X), \mathcal{G}_\lambda A(X))$. Therefore the closure of X as a subset of $(a(X), \mathcal{G}_\lambda(a(X)))$ is the same as the closure of X as a subset of $(A(X), \mathcal{G}_\lambda A(X))$, namely the Fell compactification of X. If $(X, \mathcal{G})$ is second countable, then $(\mathcal{G}, \subseteq)$ is an ω-continuous dcpo, therefore $\mathcal{G}_\sigma(\mathcal{G})$ is a second countable topology. That is, $(A(X), \sqsubseteq)$ is an ω-continuous dcpo as well (see Lemma 2.9).

Remark 5.20 *Smyth [Smy92a] uses the embedding η_X for constructing a stable compactification for X. The difference is that he restricts the range* *of η_X to $RPt(\mathcal{G})$ where $RPt(\mathcal{G}) \subseteq \mathcal{G}_\sigma(\mathcal{G})$ is the space of proximal filters of $\mathcal{G}$ and shows that $\eta_X[X]$ is dense in $RPt(\mathcal{G})$.*

Problem 5.21 *Is $RPt(\mathcal{G})$ the Fell compactification of X ? In other words, is $RPt(\mathcal{G})$ the closure of $\eta_X[X]$ in the Lawson topology of $(\mathcal{G}_\sigma(\mathcal{G}), \subseteq)$?*

5.5 Representations using maximal point hulls

The construction that we present in this section generalises the upper space and can be seen as an extension of MP-hulls and spaces of formal balls (as defined in Section 5.3). Extending Lawson's definition (Def. 5.13) we define

Definition 5.22 *Let $(X, \mathcal{G})$ be a topological space. We say that a continuous dcpo $(D, \sqsubseteq)$ is a maximal point hull (**MP-hull**) for X, if X is homeomorphic to* $\max(D)$ *as a subspace of D with respect to the relative Lawson and Scott topologies.*

We start by proving a technical lemma. Recall that $B_p = \{G \cap Q^c \mid G \in \mathcal{G}, Q \in \mathcal{Q}\}$ is a base for the patch topology, and that if $(X, \mathcal{G})$ is a sober space, then (by Theorem 3.18) $(Lens(X), \supseteq)$ is a dcpo.

Lemma 5.23 *Suppose that $(X, \mathcal{G})$ is a locally compact sober space and $L_i \in Lens(X)$ with $L_i = Q_i \cap F_i$ for $i = 1, 2$. Then*

1. *If $x \in H \in \mathcal{G}_p$, then we can find a lens $L = Q \cap F$ such that $x \in int(Q) \cap int_{cc}(F)$ and $L \subseteq H$. Hence $Lens(X)$ is a fundamental system of neighborhoods for the patch topology.*
2. *If $L_1 \ll L_2$, then there exists $L \in Lens(X)$ such that $int_p(L_1) \supseteq L$ and $int_p(L) \supseteq L_2$.*
3. *Suppose that X is coherent. Then,*
   - *$L_1 \ll L_2$ if and only if $int_p(L_1) \supseteq L_2$.*
   - *If $Q_1 \ll Q_2$ and $F_1 \ll F_2$, then $L_1 \equiv Q_1 \cap F_1 \ll Q_2 \cap F_2 \equiv L_2$.*

Proof

1. Since B_p is a base for the patch topology, there exists a patch basic open $G_1 \cap Q_1^c \in B_p$ such that $x \in G_1 \cap Q_1^c \subseteq H$. Since $x \in G_1$, then by local compactness there exists $Q \subseteq G_1$ such that $x \in int(Q)$. Similarly, since $x \in Q_1^c$, by the (wlc) property (Section 3.7) there exists $F \in \mathcal{F}$ and $Q_3 \in \mathcal{Q}$ such that $x \in Q_3^c \subseteq F \subseteq Q_1^c$. Now,

$$x \in int(Q) \cap Q_3^c \subseteq Q \cap F \subseteq G_1 \cap Q_1^c.$$

Let $L = Q \cap F$ and observe that $int(Q) \in \mathcal{G}$ and $Q_3^{\mathsf{c}} \in \mathcal{G}_{cc}$. For the second claim it suffices to notice that $int(Q) \cap Q_3^{\mathsf{c}}$ is a patch basic open set with $x \in int(Q) \cap Q_3^{\mathsf{c}} \subseteq int_p(L)$ and $L \subseteq H$.

2. The following assertions follow from parts (2) and (3) of Proposition 3.10. Since $(\mathcal{Q}\backslash\{\emptyset\}, \supseteq)$ and $(\mathcal{F}, \supseteq)$ are continuous dcpo's, there exist $\ll$-directed families $\{Q_i\}_{i\in I}$ and $\{F_j\}_{j\in J}$ in $\mathcal{Q}$ and $\mathcal{F}$ respectively such that $Q_2 = \bigcap_{\downarrow} Q_i$ and $F_2 = \bigcap_{\downarrow} F_j$. Hence

$$L_2 = \bigcap_{\downarrow\, i} Q_i \cap \bigcap_{\downarrow\, j} F_j = \bigcap_{\downarrow\, i,j} (Q_i \cap F_j).$$

Since $L_1 \ll L_2$, there must exist i_0 and j_0 such that $L_1 \sqsubseteq Q_{i_0} \cap F_{j_0}$. Since $Q_{i_0} \ll_{\mathcal{Q}} Q_2$ and $F_{j_0} \ll_{\mathcal{F}} F_2$, by the interpolation property (Proposition 1.3) we can find $Q_{i_1} \in \mathcal{Q}$ and $F_{i_1} \in \mathcal{F}$ such that $Q_{i_0} \ll_{\mathcal{Q}} Q_{i_1} \ll_{\mathcal{Q}} Q_2$ and $F_{i_0} \ll_{\mathcal{F}} F_{i_1} \ll_{\mathcal{F}} F_2$. Then, there exist further $K_{j_0}, K_{j_1} \in \mathcal{Q}$ such that,

$$\begin{aligned} L_1 &\supseteq Q_{i_0} \cap F_{j_0} & \\ &\supseteq int(Q_{i_0}) \cap K_{j_0}^{\mathsf{c}} & \text{a patch open} \\ &\supseteq Q_{i_1} \cap F_{j_1} & \text{a lens} \\ &\supseteq int(Q_{i_1}) \cap K_{j_1}^{\mathsf{c}} & \text{a patch open} \\ &\supseteq Q_2 \cap F_2 = L_2 & \end{aligned}$$

Letting $L := Q_{i_1} \cap F_{j_1}$ the claim follows.

3. By part (2) we have that $L_1 \ll L_2$ implies that $int_p(L_1) \supseteq L_2$. By part (5) of Proposition 3.16 we know that $Lens(X) \subseteq \mathcal{K}_p$. Once again the following assertions follow from parts (2)-(4) of Proposition 3.10. The topology $\mathcal{G}_p$ is Hausdorff hence $\mathcal{K}_p = \mathcal{Q}_p$ and $(X, \mathcal{G}_p)$ is sober (part (vi)-2 of Lemma 3.41). Suppose that $int_p(L_1) \supseteq L_2$ and $L_2 = \bigsqcup^{\uparrow}_{j\in J} L_j$. Since $Lens(X) \subseteq \mathcal{Q}_p$, $\{L_j\}_{j\in J}$ is a filtered family of sets in $\mathcal{Q}_p$ with $\bigcap_{\downarrow} L_j \subseteq int_p(L_1)$, by part (3) of Proposition 3.9 there exists $j_0 \in J$ with $L_{j_0} \subseteq int_p(L_1)$. It follows that $L_1 \sqsubseteq L_{j_0}$, hence $L_1 \ll L_2$.

Finally, if $Q_1 \ll Q_2$ and $F_1 \ll F_2$, then $int(Q_1) \supseteq Q_2$ and there exists $Q \in \mathcal{Q}$ with $F_1 \supseteq Q^{\mathsf{c}} \supseteq F_2$. Therefore, $L_1 \supseteq int(Q_1) \cap Q^{\mathsf{c}} \supseteq L_2$. Since $int(Q_1) \cap Q^{\mathsf{c}} \in B_p$ it follows that $int_p(L_1) \supseteq L_2$, and we have just noticed that the latter is equivalent to $L_1 \ll L_2$. ■

Remark 5.24 *The reader might find it strange that in the proof of claim (3) of Lemma 5.23 we have to use Proposition 3.9 when we already know that $Lens(X) \subseteq \mathcal{Q}_p$ and for all $Q_1, Q_2 \in \mathcal{Q}_p$,*

$$Q_1 \ll Q_2 \Leftrightarrow int(Q_1) \supseteq Q_2$$

by Proposition 3.10-(2). The reason is that the way below relation $\ll$ in Lens(X) could have been different from the way below relation in $\mathcal{Q}_p$. For example, consider $(\overline{\mathbb{N}}, \leq)$ endowed with the Scott topology. Then on the subspace $\{0, \infty\}$ we have that $\infty \ll \infty$ but this does not hold on $\overline{\mathbb{N}}$. Another (important) example of this situation occurs for the probabilistic powerdomains $\mathbf{P}^1 D$ and $\mathbf{P}D$ of a pointed continuous dcpo $(D, \sqsubseteq)$. Although $\mathbf{P}^1 D \subseteq \mathbf{P}D$, their way below relations differ in general (see Section 3 in [Eda95b]). More strange facts about the way below relation follow.

The reader might also feel tempted to prove the converse of the claim (2) in Lemma 5.23 in order to conclude that $Lens(X)$ is a continuous dcpo. In particular this would imply that $(Lens(X), \supseteq)$ is locally compact. Unfortunately this only seems to be the case when the patch topology is locally compact or when X is coherent (see Theorem 5.27 below). Hofmann and Mislove [HM81, Proposition 4.11] quote a result by Lawson that establishes sufficient conditions for the patch topology to be locally compact.

Problem 5.25 *Under which conditions is $(Lens(X), \supseteq)$ a continuous poset?*

The fact that $Lens(X)$ is not necessarily a continuous dcpo deserves a bit more of explanation. Suppose, for example, that $L \in Lens(X)$ and $\{Q_i\}_{i \in I}$ and $\{F_i\}_{i \in I}$ are directed families in $\mathcal{Q}$ and $\mathcal{F}$ respectively with

$$\begin{aligned} \uparrow L &= (\textstyle\bigcap_{\downarrow} Q_i) \\ cl(L) &= (\textstyle\bigcap_{\downarrow} F_i). \end{aligned}$$

Then by Proposition 5.2 we know that $L = \bigcap_{\downarrow}(Q_i \cap F_i)$. On the other hand if $\{L_i\}_{i \in I}$ is a directed family of lenses converging to L we know that

$$L = (\textstyle\bigcap_{\downarrow} \uparrow L_i) \cap (\textstyle\bigcap_{\downarrow} cl(L_i)),$$

but it is not necessarily the case that $\uparrow L = (\bigcap_{\downarrow} \uparrow L_i)$ and $cl(L) = (\bigcap_{\downarrow} cl(L_i))$. Notice that we are taking the canonical representatives of the lenses. Another way of looking at this, is to consider the set $\Lambda X = \{(Q, F) \in \mathcal{Q} \times \mathcal{F} \mid Q \cap F \neq \emptyset\}$ partially ordered by

$$(Q_1, F_1) \sqsubseteq_1 (Q_2, F_2) \text{ if } Q_1 \supseteq Q_2 \text{ and } F_1 \supseteq F_2.$$

That is, the inherited order from $\mathcal{Q} \times \mathcal{F}$ as a Cartesian product of continuous dcpo's (recall the statement of Proposition 5.3). It follows that $Q_1 \ll_{\mathcal{Q}} Q_2$ and $F_1 \ll_{\mathcal{F}} F_2$ if and only if $(Q_1, F_1) \ll_1 (Q_2, F_2)$. For any $L \in Lens(X)$ define

$$\widetilde{L} = \{(Q, F) \in \Lambda X \mid L = Q \cap F\}.$$

Notice that $\widetilde{L}$ has a maximum element, that is $(\uparrow L, cl(L))$. Now we want to refine the order in ΛX as follows

$$(Q_1, F_1) \sqsubseteq_2 (Q_2, F_2) \text{ if } Q_1 \cap F_1 \supseteq Q_2 \cap F_2$$

i.e., we are giving an ordering for the classes $\widetilde{L}$. Notice that $(Q_1, F_1) \sqsubseteq_1 (Q_2, F_2)$ obviously implies that $(Q_1, F_1) \sqsubseteq_2 (Q_2, F_2)$. The converse implication does not hold in general, but it does for the maximum representatives of each class $\widetilde{L}$ since

$$(\uparrow L_1, cl(L_1)) \sqsubseteq_1 (\uparrow L_2, cl(L_2)) \text{ if and only if } (\uparrow L_1, cl(L_1)) \sqsubseteq_2 (\uparrow L_2, cl(L_2)).$$

The important thing to notice is that $(\Lambda X, \sqsubseteq_2)$ is only a preordered space, not a poset. So we might want to consider the quotient space $\Lambda X/\sqsubseteq_2$, which turns out to be isomorphic to $(Lens(X), \supseteq)$, therefore the equivalence classes are precisely the sets $\widetilde{L}$ with $L \in Lens(X)$. As one might expect, strange things happen to the way below relation. The second part of Lemma 5.23 is a consequence of the interpolation property of $\ll_1$, i.e. $(Q_1, F_1) \ll_1 (Q_2, F_2)$ implies that there exists (Q', F') such that $(Q_1, F_1) \ll_1 (Q', F') \ll_1 (Q_2, F_2)$. Equivalently,

$$\widetilde{L}_1 \ll_2 \widetilde{L}_2 \text{ implies that } (\uparrow L_1, cl(L_1)) \ll_1 (\uparrow L_2, cl(L_2))$$

where $\ll_2$ is the way below relation in the quotient space $\Lambda X/\sqsubseteq_2$. For our purposes the best solution is to stay with $(\Lambda X, \sqsubseteq_1)$.

Definition 5.26 *Suppose that $(X, \mathcal{G})$ is a locally compact sober space. Define the space of* ***formal lenses*** *as*

$$\Lambda X = \{(Q, F) \in \mathcal{Q} \times \mathcal{F} \mid Q \cap F \neq \emptyset\}$$

partially ordered by $(Q_1, F_1) \sqsubseteq (Q_2, F_2)$ if $Q_1 \supseteq Q_2$ and $F_1 \supseteq F_2$.

Theorem 5.27 *Suppose that $(X, \mathcal{G})$ is a locally compact sober space. Then,*

1. *$(\Lambda X, \sqsubseteq)$ is a continuous dcpo.*
2. *If X is second countable, then ΛX is ω-continuous.*
3. *If X is coherent, then $(\Lambda X, \sqsubseteq)$ and $(Lens(X), \supseteq)$ are bounded complete domains (hence coherent).*

Proof (Similar to the proof of Theorem 3.18).

$\boxed{1}$ Since $(\mathcal{Q}, \supseteq)$, $(\mathcal{F}, \supseteq)$ are continuous dcpo's, then (by Proposition 5.3) $(\mathcal{Q} \times \mathcal{F}, \sqsubseteq)$ is a continuous dcpo where the order $\sqsubseteq$ is given by $(Q_1, F_1) \sqsubseteq$

(Q_2, F_2) if and only if $Q_1 \supseteq Q_2$ and $F_1 \supseteq F_2$.

- **Claim:** ΛX is a lower subset of $\mathcal{Q} \times \mathcal{F}$. Suppose that $(Q_1, F_1) \sqsubseteq (Q_2, F_2)$ and $(Q_2, F_2) \in \Lambda X$. Since $Q_1 \cap F_1 \supseteq Q_2 \cap F_2 \neq \emptyset$, trivially $(Q_1, F_1) \in \Lambda X$.

- **Claim:** ΛX is closed in $\mathcal{Q} \times \mathcal{F}$. It suffices to show that ΛX is closed under directed lubs. Suppose that $\langle (Q_i, F_i) \rangle_{i \in I}$ is a directed subset of ΛX. Then clearly $\langle Q_i \rangle_{i \in I}$ and $\langle F_i \rangle_{i \in I}$ are directed subsets of $(\mathcal{Q}, \supseteq)$ and $(\mathcal{F}, \supseteq)$ respectively. Let $(Q, F) = (\bigcap_{\downarrow} Q_i, \bigcap_{\downarrow} F_i)$. It is clear that if $Q \cap F \neq \emptyset$, then $\bigsqcup^{\uparrow}_{i \in I}(Q_i, F_i) = (Q, F)$. If $Q \cap F = \emptyset$, then by Lemma 3.17 there exist $h, k \in I$ such that $Q_h \cap F_k = \emptyset$. Since $\langle (Q_i, F_i) \rangle_{i \in I}$ is directed we can find $j \in I$ with $(Q_h, F_h), (Q_k, F_k) \sqsubseteq (Q_j, F_j)$. Which implies that $Q_j \cap F_j = \emptyset$: a contradiction since $(Q_j, F_j) \in \Lambda X$. Therefore $(Q, F) \in \Lambda X$ and the claim follows.

Since ΛX is a closed subset of the continuous dcpo $\mathcal{Q} \times \mathcal{F}$, it is a continuous dcpo.

$\boxed{2}$ By part (2) of Lemma 3.11 we know that if $(X, \mathcal{G})$ is second countable, then the dcpo's $(\mathcal{F}, \supseteq)$ and $(\mathcal{Q} \backslash \{\emptyset\}, \supseteq)$ are ω-continuous; hence by Proposition 5.3, $\mathcal{Q} \times \mathcal{F}$, and hence ΛX, are also ω-continuous.

$\boxed{3}$ Suppose that X is coherent. If $(Q_1, F_1), (Q_2, F_2) \sqsubseteq (Q, F) \in \Lambda X$, then $(Q_1 \cap Q_2, F_1 \cap F_2) \in \Lambda X$ and clearly

$$(Q_1 \cap Q_2, F_1 \cap F_2) = (Q_1, F_1) \vee (Q_2, F_2).$$

Which shows that ΛX is bounded complete and hence a coherent domain by Proposition 5.4. By Theorem 3.18 we know that $(Lens(X), \supseteq)$ is a bounded complete dcpo; it remains to check it is continuous.

We claim that for every $L \in Lens(X)$ the set $\twoheaddownarrow L$ is directed. Suppose that $L_i \ll L$ for $i = 1, 2$. Then by part (3) of Lemma 5.23 we have that $int_p(L_i) \supseteq L$. Hence, for $i = 1, 2$ we have

$$\begin{aligned} L_i &\supseteq L_1 \cap L_2 \\ &\supseteq int_p(L_1 \cap L_2) \\ &= int_p(L_1) \cap int_p(L_2) \\ &\supseteq L. \end{aligned}$$

Therefore $L_i \sqsubseteq (L_1 \cap L_2) \ll L$ with $(L_1 \cap L_2) \in Lens(X)$ for $i = 1, 2$.

Finally suppose that $L := (Q \cap F) \in Lens(X)$. Since $(\mathcal{Q} \backslash \{\emptyset\}, \supseteq)$ and $(\mathcal{F}, \supseteq)$ are continuous dcpo's, there exist $\ll$-directed families $\langle Q_i \rangle_{i \in I}$ and $\langle F_j \rangle_{j \in J}$ in

$\mathcal{Q}$ and $\mathcal{F}$ respectively such that $Q = \bigcap_{\downarrow} Q_i$ and $F = \bigcap_{\downarrow} F_j$. Since $Q_i \ll_{\mathcal{Q}} Q$ and $F_j \ll_{\mathcal{F}} F$, by part (3) of Lemma 5.23 we have that $Q_i \cap F_j \ll L$ for all i, j. Now,

$$\begin{aligned} L &= Q \cap F \\ &= (\bigcap_{i}{}_{\downarrow} Q_i) \cap (\bigcap_{j}{}_{\downarrow} F_j) \\ &= \bigcap_{i,j}{}_{\downarrow} (Q_i \cap F_j). \end{aligned}$$

It follows that $L = \bigsqcup^{\uparrow}_{(i,j) \in I \times J} (Q_i \cap F_j)$ with $Q_i \cap F_j \ll L$ for all i, j. Therefore $Lens(X)$ is a bounded complete domain. ■

Let the function $e : X \to \Lambda X$ be defined as $e(x) = (sat(\{x\}), cl(x)) = (\uparrow x, \downarrow x)$ and recall that $\widetilde{x}$ abbreviates $e(x)$.

Theorem 5.28 *Suppose that $(X, \mathcal{G})$ is a locally compact sober space. Then*

1. *The function $e : X \to (\Lambda X)^{\lambda}$ is a subspace embedding with $e[X] = \max(\Lambda X)$.*
2. *$(\Lambda X, \sqsubseteq)$ is an MP-hull for $(X, \mathcal{G}_p)$*
3. *If X is coherent, then $(Lens(X), \supseteq)$ is also an MP-hull for $(X, \mathcal{G}_p)$.*

Proof To avoid confusion we distinguish the operators $\Uparrow$ and $\uparrow$ on X from the operators $\Uparrow_{\Lambda}$ and $\uparrow_{\Lambda}$ on $(\Lambda X, \sqsubseteq)$. In the following, (Q, F) denotes any element of $(\Lambda X, \sqsubseteq)$.

1. **Claim:** $e[X] = \max(\Lambda X)$. $\boxed{\subseteq}$ Suppose that $\widetilde{x} = (\uparrow x, \downarrow x) \sqsubseteq (Q, F)$. Then $\uparrow x \supseteq Q$ and $\downarrow x \supseteq F$ and it follows that $\{x\} = \uparrow x \cap \downarrow x \supseteq Q \cap F$. Since $Q \cap F$ is nonempty by assumption, then $Q \cap F = \{x\}$. Therefore $\uparrow x = Q$ and $\downarrow x = F$, i.e. $\widetilde{x} \in \max(\Lambda X)$. $\boxed{\supseteq}$ Suppose that $(Q, F) \in \max(\Lambda X)$. Since $Q \cap F \neq \emptyset$ there must exist $x \in Q \cap F$. Then $\uparrow x \subseteq Q$ and $\downarrow x \subseteq F$, i.e. $(Q, F) \sqsubseteq \widetilde{x}$. Since we assumed that (Q, F) is maximal, then $(Q, F) = \widetilde{x}$.

2. **Claim:** e is continuous with respect to the Scott topology in ΛX. Since the set $\{\Uparrow_{\Lambda}(Q, F) \mid (Q, F) \in \Lambda X\}$ is a base for the Scott topology, it suffices to show that $e^{-1}(\Uparrow_{\Lambda}(Q, F)) \in \mathcal{G}_p$ for all $(Q, F) \in \Lambda X$. Now $e^{-1}(\Uparrow_{\Lambda}(Q, F)) = \{x \in X \mid (Q, F) \ll \widetilde{x}\}$. But $(Q, F) \ll \widetilde{x}$ if and only if $Q \ll_{\mathcal{Q}} \uparrow x$ and $F \ll_{\mathcal{F}} \downarrow x$. Notice that $Q \ll_{\mathcal{Q}} \uparrow x$ is equivalent to $x \in int(Q)$. While $F \ll_{\mathcal{F}} \downarrow x$ if and only if there exists $Q_1 \in \mathcal{Q}$ such that $F \supseteq Q_1^{\complement} \supseteq \downarrow x$, that is, $x \in int_{cc}(F)$. Therefore $e^{-1}(\Uparrow_{\Lambda}(Q, F)) = int(Q) \cap int_{cc}(F) \in \mathcal{G}_p$.

3. **Claim:** e is continuous with respect to the lower topology in ΛX. For all $(Q,F) \in \Lambda X$ we have that $e^{-1}(\uparrow_\Lambda(Q,F)) = \{x \in X \mid (Q,F) \sqsubseteq \widetilde{x}\}$. Notice that $(Q,F) \sqsubseteq \widetilde{x} \Leftrightarrow (\uparrow x \subseteq Q \text{ and } \downarrow x \subseteq F) \Leftrightarrow x \in Q \cap F$. Therefore $e^{-1}(\uparrow_\Lambda(Q,F)) = Q \cap F$; a patch closed set. Since $\{\bigcup_{k=1}^n \uparrow_\Lambda I_k \mid I_k \in \Lambda X,\ k \in \mathbb{N}\}$ is a base for the closed sets of the lower topology in ΛX, the claim follows.

4. **Claim:** e is continuous with respect to the Lawson topology in ΛX. Immediate from the previous claims since the Lawson topology has as a subbase the union of the Scott and lower topologies.

5. **Claim:** $e : X \to \max(\Lambda X)$ is an open map. Let G be a patch open set of X. Define the Scott open set $\Box G$ on ΛX as $\Box G = \bigcup\{\twoheaduparrow_\Lambda(Q,F) \mid Q \cap F \subseteq G \text{ and } (Q,F) \in \Lambda X\}$. Suppose that $x \in G$, then by part (1) of Lemma 5.23 we can find $Q \cap F \in \mathit{Lens}(X)$ such that $x \in \mathit{int}(Q) \cap \mathit{int}_{cc}(F)$ and $Q \cap F \subseteq G$. It follows that $Q \ll_{\mathcal{Q}} \uparrow x$ and $F \ll_{\mathcal{F}} \downarrow x$, i.e. $(Q,F) \ll \widetilde{x}$. This shows that $\widetilde{x} \in \Box G \cap \max(\Lambda X)$. Conversely if $\widetilde{x} \in \Box G \cap \max(\Lambda X)$, then there exist $(Q,F) \in \Lambda X$ such that $Q \cap F \subseteq G$ and $(Q,F) \ll \widetilde{x}$. Then $(Q,F) \sqsubseteq \widetilde{x}$ and $x \in Q \cap F$, which implies $x \in G$. We conclude that $e[G] = \Box G \cap \max(\Lambda X)$ is an open subset of $\max(\Lambda X)$.

Since X is a T_0 space, then e is clearly a one-to-one function. Therefore e is a subspace embedding and parts (1) and (2) follow. For the third part define $e' : X \to \mathit{Lens}(X)$ as $e'(x) = \{x\}$. We observed in Theorem 3.18, that $e'[X] = \max(\mathit{Lens}(X))$ and by part (3) of Theorem 5.27 we know that $\mathit{Lens}(X)$ is a coherent domain. Let $\uparrow_{Le}$ and $\twoheaduparrow_{Le}$ denote the operators $\uparrow$ and $\twoheaduparrow$ in the space of lenses. In the light of Lemma 5.23, since $\mathit{Lens}(X)$ is fundamental system of neighborhoods for $\mathcal{G}_p$, it is not difficult to see that for $L \in \mathit{Lens}(X)$ we have that $e'[L] = \max(\mathit{Lens}(X)) \cap \uparrow_{Le} L$ and $e'[\mathit{int}_p(L)] = \max(\mathit{Lens}(X)) \cap \twoheaduparrow_{Le} L$. It should be clear that $(\mathit{Lens}(X), \supseteq)$ is an MP-hull for $(X, \mathcal{G}_p)$. Alternatively the reader can verify that the proof above *mutatis mutandis* (substituting (Q,F) by $Q \cap F$) also gives us the desired conclusion. ■

In other words, the last theorem says that:

- The patch space of a locally compact sober space can be realised as the subspace of maximal elements of a continuous dcpo (recall that continuous dcpo's are, in particular, locally compact sober spaces).

- Furthermore if the space is stably locally compact, then the continuous dcpo can be chosen to be coherent (recall that coherent domains are, in particular, stably locally compact spaces).

- If the space is compact ($X \in \mathcal{Q}$), then the MP-hull has a least element since $(X, X) \in \Lambda X$ and $X \in Lens(X)$.
- If the space is second countable, then the MP-hull is ω-continuous.

Corollary 5.29 *Suppose that $(X, \mathcal{G})$ is a locally compact sober space. Then $(\Lambda X, \sqsubseteq, e)$ is a dense S-domain representation for $(X, \mathcal{G}_p)$.*

Notice that for every $G \in \mathcal{G}_p$ the set $\Box G = \bigcup\{\Uparrow_\Lambda(Q, F) \mid Q \cap F \subseteq G\}$ is Scott open in $(\Lambda X, \sqsubseteq)$ and $e[G] = \max(\Lambda X) \cap \Box G$. Conversely, if H is Scott open in $(\Lambda X, \sqsubseteq)$, then the set $\nabla H = \bigcup\{int(Q) \cap int_{cc}(F) \mid (Q, F) \in H\} \in \mathcal{G}_p$ and $e[\nabla H] = \max(\Lambda X) \cap H$. This shows that $(\Lambda X, \sqsubseteq)$ is a model for $(X, \mathcal{G}_p)$ in the sense of Martin [Mar98]. As happens with the representation in Section 5.3.3 we have that $G = \nabla \Box G$ and $H \subseteq \Box \nabla H$.

Corollary 5.30 *Let $(X, \mathcal{G})$ be a T_0 topological space. Then there exists a $T_{3\frac{1}{2}}$ (Hausdorff) topology $\mathcal{H}$ on X such that $\mathcal{G} \subseteq \mathcal{H}$ and $(X, \mathcal{H})$ embeds as a subspace of* $\max(D)$ *where $(D, \sqsubseteq)$ is a continuous dcpo equipped with the Lawson topology.*

Proof Recall from Section 5.3.1 that $(X, \mathcal{G})$ embeds as a subspace of the stably compact space $(\mathbb{S}X, \mathcal{G}_\sigma\mathbb{S})$. By the last theorem $(\mathbb{S}X, \mathcal{G}_\lambda\mathbb{S})$ has an MP-hull $(\Lambda(\mathbb{S}X), \sqsubseteq)$. Let $\mathcal{H}$ be the subspace topology induced on X by the Lawson topology $\mathcal{G}_\lambda\mathbb{S}$ on $\mathbb{S}X$. Then $(X, \mathcal{H})$ is completely regular since $(\mathbb{S}X, \mathcal{G}_\lambda\mathbb{S})$ is compact Hausdorff. ■

As in part (2) of Proposition 5.14, Theorem 5.28 has the following converse.

Theorem 5.31 *Suppose that $(D, \sqsubseteq)$ is a continuous dcpo, $X \subseteq D$ and the relative Scott and Lawson topologies agree on X. That is, $\mathcal{G}_\lambda D \restriction_X = \mathcal{G}_\sigma D \restriction_X$. Let $\mathcal{G} := \mathcal{G}_\lambda D \restriction_X$. Then*

1. *$(X, \mathcal{G})$ is $T_{3\frac{1}{2}}$ (Hausdorff).*
2. *If X is closed in D^λ and D is stably compact, then $(X, \mathcal{G})$ is compact Hausdorff.*
3. *If X is closed in D^λ and D is coherent, then $(X, \mathcal{G})$ is locally compact Hausdorff.*

Proof Recall from Proposition 1.10 that a continuous dcpo is locally compact sober and the patch topology agrees with the Lawson topology. Hence, by part (4) of Proposition 3.10, D^λ is $T_{3\frac{1}{2}}$ and any subspace of a $T_{3\frac{1}{2}}$ space is $T_{3\frac{1}{2}}$ [Dug66, Theorem VII-7.2]. This shows the first part. The second part is obvious since D^λ is compact Hausdorff. For the third part suppose that $G \in \mathcal{G}$ and $x \in G$. Then $G = H \cap X$ for some $H \in \mathcal{G}_\sigma D$. Since D is a continuous dcpo, there exists $y \in H$ with $y \ll x$. Now notice that $\uparrow y \in \mathcal{Q}_\sigma D$. But since D is coherent, then by part (2) of Lemma 3.12 we have that $\mathcal{Q}_\sigma D \subseteq \mathcal{K}_\lambda D$. That is, $\uparrow y$ is compact in D^λ. Since D^λ is Hausdorff and X is closed in D^λ, then $(\uparrow y \cap X) \in \mathcal{K}_\lambda D$. It follows that $(\uparrow y \cap X)$ is compact in $\mathcal{G}$. Hence we have shown that

$$x \in (\twoheaduparrow y \cap X) \subseteq (\uparrow y \cap X) \subseteq G,$$

where $(\twoheaduparrow y \cap X) \in \mathcal{G}$ and $(\uparrow y \cap X) \in \mathcal{K}X$. The conclusion follows. ■

Problem 5.32 *Suppose that* $(D, \sqsubseteq)$ *is a coherent domain and* $\mathcal{G}_\lambda D \restriction_{\max(D)} = \mathcal{G}_\sigma D \restriction_{\max(D)}$. *Is* $\max(D)$ *with the subspace topology a locally compact Hausdorff space? Notice that by the previous theorem it suffices to show that* $\max(D)$ *is Lawson closed as it happens for the upper space (see Remark 5.11). What happens if we assume that* $(D, \sqsubseteq)$ *is bounded complete?*

The reader should pause here and compare the information gathered from Theorems 3.18, 5.27, 5.28 and Lemma 5.23 against the statements of Propositions 5.8, 5.12, 5.14 and 5.16.

Remark 5.33 *If* $(X, \mathcal{G})$ *is a locally compact Hausdorff space, then* $(Lens(X), \supseteq) = (\mathbf{U}X, \supseteq)$. *That is, the space of lenses agrees with the upper space.*

Proof For Hausdorff spaces we have that $\mathcal{K} = \mathcal{Q}$, and if $F \in \mathcal{F}$ and $K \in \mathcal{K}$, then $F \cap K \in \mathcal{K}$. Noticing that $X \in \mathcal{F}$, it follows that $Lens(X) = \mathcal{K}$. Since $\mathcal{K} \subseteq \mathcal{F}$ then $\mathcal{G}_p = \mathcal{G}$. ■

Example 5.34 *Consider* $[0,1]$ *with its usual metric topology. Then* $(Lens\ [0,1], \supseteq)$ *is the upper space of* $[0,1]$. *If we consider* $([0,1], \leq)$ *as a stably compact space with the Scott topology, then* $(Lens[0,1], \supseteq)$ *is the subset of closed intervals of* $[0,1]$ *ordered by reverse inclusion (see Section 8.3 below).*

6. Domain theoretic integration and interval integrals

6.1 Summary

In this chapter we unify the two main approaches to the construction of integrals using domain theory, namely the Riemann-Edalat integral and the valuation integral. The Riemann-Edalat (R-) integral was originally introduced by Edalat in [Eda95b] as a generalisation of the Riemann integral for compact metric spaces. Later it was extended by Edalat and Negri [EN98] to locally compact Hausdorff spaces, and recently by Howroyd [How00] to any Hausdorff space $(X, \mathcal{G})$ which admits a pointed S-domain representation (Definition 5.1). The valuation integral is defined for lower semi-continuous functions with respect to any continuous valuation on a topological space. It comprises the work of Jones and Plotkin [Jon89][JP89], Kirch [Kir93], Tix [Tix95], and Heckmann [Hec96].

While the R-integral is defined for any bounded real-valued function with respect to a τ-smooth probability measure, the valuation integral exists for non-negative (possibly infinitely valued) lower semi-continuous functions with respect to any continuous valuation. Suppose that X is a Hausdorff space which admits a pointed S-domain representation $(D, \sqsubseteq, e, \perp)$. Then every τ-smooth probability measure μ on X, induces (via e) a continuous valuation μ_e on D (see Section 6.2.3 below). The R-integral uses simple valuations way below μ_e (recall the statement of Proposition 2.5) to construct upper and lower approximations to the integral of a bounded function $f : X \to \mathbb{R}$. On the other hand, the valuation integral of a lower semi-continuous function $F : D \to \overline{\mathbb{R}}_+$ with respect to μ_e is approximated by the integrals of F with respect to the simple valuations way below μ_e. These observations suggest that there must exist a strong relation between the integrals, nevertheless both approaches had remained unconnected so far.

The main result of the chapter (Theorem 6.40) establishes very precisely the connection between the integrals. The general idea is as follows. We restrict ourselves to bounded real-valued functions and finite valuations and show that we can construct upper and real-valued versions of the valuation integral (Section 6.5). Any bounded real-valued function f on a topological space X can be sandwiched between its upper and lower semi-continuous envelopes f_E and f^E. If X embeds as a subspace of a continuous dcpo D, these envelopes can be extended to lower and upper semi-continuous functions f_D and f^D whose domain is D. The lower and upper valuation integrals of f_D and f^D with respect to μ_e agree with the lower and upper R-integrals of f with respect to μ. Moreover, they also agree with the Lebesgue integrals $\mu(f_E)$, $\overline{\mu}_e(f_D)$ and $\mu(f^E)$, $\overline{\mu}_e(f^D)$ where $\overline{\mu}_e$ is the Borel measure which extends μ_e. We finalise the chapter by showing that these constructions extend with almost no modification to functions taking values on the domain of intervals $\mathbf{I}\overline{\mathbb{R}}$, giving thus interval valued versions of the above integrals.

It should be mentioned that Lawson and Lu [LL00] have recently developed a domain theoretic S-integral using contents (i.e., finitely additive measures). This notion of integral agrees with the R-integral for spaces which admit an MP-hull (Definition 5.22). On a recent talk they have also indicated that the S-integral agrees with the valuation integral in topological spaces, giving thus an alternative unification theory. The latter claim has not been published yet, but we can foresee some overlapping with some of the results in this chapter.

Most of the chapter consists in preliminary material. This is justified because in subsequent chapters we will use these results for discussing weak topologies in spaces of measures.

6.2 Auxiliary material

Suppose that $(P, \sqsubseteq)$ is a poset. Recall the following facts and definitions. The lower topology, $\mathcal{G}_l P$ is the topology generated by the sets $\{(\uparrow x)^{\mathsf{c}} \mid x \in P\}$. The Lawson topology is defined as $\mathcal{G}_\lambda P := \mathcal{G}_\sigma P \vee \mathcal{G}_l P$. For $\bullet = \sigma, l, \lambda$ we set $P^\bullet = (P, \mathcal{G}_\bullet P)$. Recall from Definition 5.1 that an S-domain representation for a space $(X, \mathcal{G})$ is a continuous dcpo $(D, \sqsubseteq)$ where $(X, \mathcal{G})$ embeds as a subspace of $(D, \mathcal{G}_\sigma D)$. Consider any $[a, b] \subseteq \overline{\mathbb{R}} \equiv [-\infty, \infty]$. Then $([a, b], \leq)$ is a continuous lattice and $[a, b]^{\leq}$ is another notation for $[a, b]^\sigma$. Similarly $[a, b]^{\geq}$ denotes the continuous lattice $([a, b], \leq^{op})$, topologised with the Scott topology; where $x \leq^{op} y$ if and only if $x \geq y$. By $[a, b]^\lambda$ we denote the interval $[a, b]$ with its usual metric topology.

- $f \in [X \to [a, b]^{\leq}]$ if and only if f is lower semi-continuous (LSC) on $[a, b]$.
- $f \in [X \to [a, b]^{\geq}]$ if and only if f is upper semi-continuous (USC) on $[a, b]$.

For a bounded function $f : X \to \mathbb{R}$, we fix a $M_f \in \mathbb{R}$ such that $f(X) \subseteq [-M_f, M_f]$. When we write $f : X \to [-M_f, M_f]$, it is understood that f is bounded hence $M_f < \infty$ exists. We define

$$\lceil X \to [a,b]^{\leq} \rceil := \{f \in [X \to [a,b]^{\leq}] \mid f \text{ is bounded}\}$$

and similarly $\lceil X \to [a,b]^{\geq} \rceil$. Accordingly $\lceil X \to \mathbb{R}^{\leq} \rceil$ and $\lceil X \to \mathbb{R}^{\geq} \rceil$ denote the sets of bounded LSC and USC real valued functions.

6.2.1 Upper and lower semicontinuous envelopes

For the rest of this subsection $[a,b]$ denotes any closed (possibly unbounded) subinterval of $\overline{\mathbb{R}} \equiv [-\infty, \infty]$.

Lemma 6.1 *Let $(X, \mathcal{G})$ be any topological space and $f : X \to [a,b]$ be any function. Then there exists a greatest LSC function $f_E : X \to [a,b]$ such that $f_E \leq f$ given by*

$$f_E(x) := \sup\{g(x) \mid g \in [X \to [a,b]^{\leq}] \textit{ and } g \leq f\} \textit{ for all } x \in X.$$

Proof Notice that the set $\{g \in [X \to [a,b]^{\leq}] \mid g \leq f\}$ is nonempty since the constant function $x \mapsto a$ belongs to it. Now choose any $c \in [a,b]$ and suppose that $f_E(x) \in (c,b]$. Then there exists $g \in [X \to [a,b]^{\leq}]$ such that $c < g(x) \leq f_E(x)$. It follows that $x \in g^{-1}(c,b]$. Conversely if there exists $g \in [X \to [a,b]^{\leq}]$ with $g \leq f$ and $g(x) \in (c,b]$, then $f_E(x) \in (c,b]$. So we can write

$$f_E^{-1}(c,b] = \bigcup\{g^{-1}(c,b] \mid g \in [X \to [a,b]^{\leq}] \text{ and } g \leq f\}.$$

This shows that $f_E^{-1}(c,b]$ is an open set for all $c \in [a,b]$. Since $f_E^{-1}[a,b] = X$ the conclusion follows. ■

Corollary 6.2 *Let $(X, \mathcal{G})$ be any topological space and $f : X \to [a,b]$ be any function. Then there exists a minimum USC function $f^E : X \to [a,b]$ such that $f \leq f^E$ given by*

$$f^E(x) := \inf\{g(x) \mid g \in [X \to [a,b]^{\geq}] \textit{ and } f \leq g\} \textit{ for all } x \in X.$$

f_E and f^E are called the **lower and upper semi-continuous envelopes** of f. Let D_f denote the set of discontinuity points of f with respect to the metric topology of $\mathbb{R}$. Then D_f can be expressed as:

$$\begin{aligned} D_f &= \{x \mid f_E(x) < f^E(x)\} \\ &= \textstyle\bigcup_{n \in \mathbb{N}} \{x \mid (f^E - f_E)(x) \geq \frac{1}{n}\} \end{aligned}$$

Hence D_f is a F_σ-set (in particular a Borel set) even when f is not Borel measurable. The following classical result will be useful.

Lemma 6.3 *Let $(X, \mathcal{G})$ be a topological space and $f \in [X \to [0,\infty]^{\leq}]$. Then there exists an increasing sequence of simple functions (f_n) in $[X \to [0,\infty]^{\leq}]$ with $f = \sup_{n\in\mathbb{N}} f_n$.*

Proof Define $f_0 := 0$. For all $n \geq 1$ and $0 \leq j \leq n2^n$ let $G_n^j = f^{-1}(\frac{j}{2^n}, \infty\dot{]}$. Since f is LSC, then G_n^j is open. Now define

$$f_n := \textstyle\sum_{j=1}^{n2^n} \frac{1}{2^n} \mathbb{I}_{G_n^j}$$

Since for all $j < n2^n$ we have that $G_n^j \backslash G_n^{j+1} = f^{-1}(\frac{j}{2^n}, \frac{j+1}{2^n}]$ and $G_n^{n2^n} = f^{-1}(n, \infty]$, the set $\{G_n^j \backslash G_n^{j+1} \mid j = 0, \ldots, n2^n - 1\} \cup \{G_n^{n2^n}\}$ is a partition of $f^{-1}(0,\infty]$. For $x \in G_n^j \backslash G_n^{j+1}$ and $j < n2^n$ we have that $f_n(x) = \sum_{k=1}^{j} \frac{1}{2^n} = \frac{j}{2^n} < f(x) \leq \frac{j+1}{2^n}$. For $x \in G_n^{n2^n}$ we have $f_n(x) = n < f(x)$. It follows that $f_n \sqsubseteq f$ for all $n \in \mathbb{N}$. A similar argument shows that $f_n \sqsubseteq f_{n+1}$, since for all $j < n2^n$

$$\begin{array}{rcl} G_n^j \backslash G_n^{j+1} & = & f^{-1}(\frac{j}{2^n}, \frac{j+1}{2^n}] \\ & = & f^{-1}(\frac{2j}{2^{n+1}}, \frac{2j+1}{2^{n+1}}] \cup f^{-1}(\frac{2j+1}{2^{n+1}}, \frac{2j+2}{2^{n+1}}] \\ & = & (G_{n+1}^{2j} \backslash G_{n+1}^{2j+1}) \cup (G_{n+1}^{2j+1} \backslash G_{n+1}^{2j+2}). \end{array}$$

Notice that the functions f_n are Scott continuous since the sets G_n^j are open and $[X \to [0,\infty]^{\leq}]$ is closed under finite sum and non-negative scalar product. It is straightforward to verify that $f = \sup_{n\in\mathbb{N}} f_n$. If f is bounded, then $|f(x) - f_n(x)| < \frac{1}{2^n}$ for all $x \in X$ and $n \geq M_f$. Thus the convergence is uniform. ■

6.2.2 Injective spaces and Scott's formula

Suppose that $(X, \mathcal{G})$ is a T_0 topological space. A space L is called **injective** if for every continuous function $f : X \to L$ and every topological space $(Y, \mathcal{H})$ such that X embeds as a subspace of Y, there exists a continuous function from Y to L which extends f. The next result shows that injective T_0 spaces are precisely the continuous lattices.

Proposition 6.4 *A T_0 space $(L, \mathcal{J})$ is injective if and only if $(L, \sqsubseteq_s)$ is a continuous lattice and $\mathcal{J}$ agrees with the Scott topology on L. That is, $\mathcal{J} = \mathcal{G}_\sigma L$.*

Proof See Theorem II-3.8 in [GHK+80]. ■

Moreover, suppose that $e : X \to Y$ is a subspace embedding. Then there exists a greatest continuous extension $f_e : Y \to L$ of f along e given by **Scott's formula** (see Exercise 3.14 in [GHK+80] or [Esc98]):

$$\begin{aligned} f_e(y) &:= \bigvee\nolimits^{\uparrow}\{\bigwedge\{f(x) \mid x \in e^{-1}G\} \mid G \in \mathcal{N}(y)\} \\ &= \bigvee\nolimits^{\uparrow}\{\bigwedge f[e^{-1}G] \mid y \in G \in \mathcal{H}\} \end{aligned}$$

for all $y \in Y$. In particular if Y is a continuous dcpo with the Scott topology we get:

Lemma 6.5 *Suppose that $(X, \mathcal{G})$ is a T_0 topological space and the continuous dcpo $(D, \sqsubseteq, e)$ is an S-domain representation for X. Then for every $f \in [X \to L]$ the greatest continuous extension of f along e is given by*

$$f_e(y) = \bigvee\nolimits^{\uparrow}\{\bigwedge f[e^{-1}{\Uparrow}z] \mid z \ll y \text{ and } z \in D\}$$

Proof Let $g(y) := \bigvee^{\uparrow}\{\bigwedge f[e^{-1}{\Uparrow}z] \mid z \ll y \text{ and } z \in D\}$. Now observe that for all $z \ll y$ we have that $y \in {\Uparrow}z$ and ${\Uparrow}z \in \mathcal{G}_\sigma D$. Therefore,

$$\{\bigwedge f[e^{-1}{\Uparrow}z] \mid z \ll y \text{ and } z \in D\} \subseteq \{\bigwedge f[e^{-1}G] \mid y \in G \in \mathcal{G}_\sigma D\}$$

it follows that $g(y) \le f_e(y)$ for all $y \in D$. Conversely, fix $y \in D$ and suppose that $y \in G$ with $G \in \mathcal{G}_\sigma D$. Then there exists $z \in G$ with $z \ll y$ and

$$\begin{aligned} {\Uparrow}z \subseteq G &\Rightarrow e^{-1}{\Uparrow}z \subseteq e^{-1}G \\ &\Rightarrow f[e^{-1}{\Uparrow}z] \subseteq f[e^{-1}G] \\ &\Rightarrow \bigwedge f[e^{-1}G] \le \bigwedge f[e^{-1}{\Uparrow}z] \\ &\Rightarrow f_e(y) \le g(y) \end{aligned}$$

and the conclusion follows. ■

Suppose that $(X, \mathcal{G})$ and $(Y, \mathcal{H})$ are topological spaces, $e \in [X \to Y]$ is a subspace embedding and $f : X \to [a, b]$ is any function. Then from Section 6.2.1 we know that $f_E \in [X \to [a, b]^{\le}]$, $f^E \in [X \to [a, b]^{\ge}]$ and $f_E \le f \le f^E$. Since $[a, b]^{\le}$ and $[a, b]^{\ge}$ are continuous lattices we can define $f_Y := (f_E)_e$ and $f^Y := (f^E)_e$. That is,

$$\begin{aligned} f_Y(y) &:= \sup\{\inf f_E[e^{-1}G] \mid y \in G \text{ and } G \in \mathcal{H}\} \\ f^Y(y) &:= \inf\{\sup f^E[e^{-1}G] \mid y \in G \text{ and } G \in \mathcal{H}\} \end{aligned}$$

Thus, $f_Y \in [Y \to [a, b]^{\le}]$ and $f^Y \in [Y \to [a, b]^{\ge}]$.

Remark 6.6 ***Warning!*** *Although $f_E \le f^E$ it does not follow that $f_Y \le f^Y$ unless X is a dense subspace of Y. This is because $\inf \emptyset = b$ and $\sup \emptyset = a$. For example, let $X = \{0\}$, $Y = \mathbb{R}$ and $e : X \to Y$ be given by $e(0) := 0$. Let $f : X \to [0, 1]$ be defined as $f(0) = \frac{1}{2}$. Then $f_E = f^E = f$ and*

$$f_Y(y) = \begin{cases} \frac{1}{2} & \textit{if } y = 0 \\ 1 & \textit{otherwise} \end{cases}$$

$$f^Y(y) = \begin{cases} \frac{1}{2} & \textit{if } y = 0 \\ 0 & \textit{otherwise} \end{cases}$$

Combining the previous definitions with Lemma 6.5 we obtain

Corollary 6.7 *Suppose that $(X, \mathcal{G})$ is a T_0 topological space and $(D, \sqsubseteq, e)$ is an S-domain representation for X. Then for every $f : X \to [a, b]$ the functions $f_D \in [D \to [a, b]^{\leq}]$ and $f^D \in [D \to [a, b]^{\geq}]$ given by*

$$\begin{aligned} f_D(y) &:= \sup\{\inf f_E[e^{-1}{\uparrow\!\!\!\uparrow} z] \mid z \ll y \textit{ and } z \in D\} \\ f^D(y) &:= \inf\{\sup f^E[e^{-1}{\uparrow\!\!\!\uparrow} z] \mid z \ll y \textit{ and } z \in D\} \end{aligned}$$

are the greatest[1] *lower and upper semi-continuous extensions of f_E and f^E to D along e.*

For more information on injective spaces the reader is referred to [Esc98] for instance.

6.2.3 Restriction and extension of continuous valuations on subspaces

The following remark collects some obvious facts to make them explicit. Recall that if $e : X \to Y$ is a subspace embedding for every $B \subseteq X$ and $x \in X$, we set $\widetilde{B} = e[B]$ and $\widetilde{x} = e(x)$.

Remark 6.8 *Suppose that $(X, \mathcal{G})$ and $(Y, \mathcal{H})$ are topological spaces and there exists a subspace embedding $e : X \to Y$. Then,*

1. *$\mathcal{B}X = \sigma(e)$, where $\sigma(e) = \{e^{-1}F \mid F \in \mathcal{B}Y\}$.*
2. *$\mathcal{B}Y \restriction_{\widetilde{X}} = \{\ \widetilde{E} \mid E \in \mathcal{B}X\}$*
3. *$(X, \mathcal{B}X)$ and $(\ \widetilde{X}, \mathcal{B}Y \restriction_{\widetilde{X}})$ are isomorphic measurable spaces.*

Proof

1. Since e is a continuous function, it is Borel measurable. Then $\sigma(e)$ is a σ-algebra and $\sigma(e) \subseteq \mathcal{B}X$. Since e is an embedding, for every $G \in \mathcal{G}$ there exists $H \in \mathcal{H}$ such that $G = e^{-1}H$. It follows that $\mathcal{G} \subseteq \sigma(e)$. Hence $\mathcal{B}X \subseteq \sigma(e)$ since $\mathcal{B}X$ is the minimum σ-algebra that contains $\mathcal{G}$.
2. Note that

[1] f_D with respect to the order $\leq$ and f^D with respect to the order $\geq$.

$$\begin{aligned}
\mathcal{B}Y \restriction_{\widetilde{X}} &= \{F \cap \widetilde{X} \mid F \in \mathcal{B}Y\} && \text{by definition} \\
&= \{e[e^{-1}F] \mid F \in \mathcal{B}Y\} \\
&= \{\widetilde{E} \mid E \in \mathcal{B}X\} && \text{by part (1)}
\end{aligned}$$

3. Is immediate from parts (1) and (2) since $e : X \to \widetilde{X}$ is a bijection. ■

Corollary 6.9 *Suppose that $(X, \mathcal{G})$ is a topological space and $(D, \sqsubseteq, e)$ is an S-domain representation for X. Suppose that ν is a finite continuous valuation on X. Then*

1. *The function ν_e given by $\nu_e(H) := \nu(e^{-1}H)$ for all $H \in \mathcal{G}_\sigma D$ is a finite continuous valuation on D.*

Let $\overline{\nu}_e$ denote the unique extension of ν_e to a finite, τ-smooth, perfect Borel measure on $(D, \mathcal{B}_\sigma D)$ given by Corollaries 2.22 and 4.16.

2. *The outer restriction $(\overline{\nu}_e)^*$ of $\overline{\nu}_e$ to $\widetilde{X}$ determines a finite Borel measure ν_r on X given by $\nu_r(E) := (\overline{\nu}_e)^*(\widetilde{E})$ for all $E \in \mathcal{B}X$, such that*

   $\nu_r(G) \leq \nu(G)$ for all $G \in \mathcal{G}$.

3. *If ν extends to a (τ-smooth) Borel measure $\overline{\nu}$ on $(X, \mathcal{B}X)$, then*

   - *$\overline{\nu}_e = (\overline{\nu})_e$, where $(\overline{\nu})_e$ is the measure induced by e on D.*
   - *$\nu_r = \overline{\nu}$.*

Proof

1. This is immediate since e is a continuous function.

2. Recall from Proposition 1.18 that the outer restriction $(\overline{\nu}_e)^*$ is a measure on $\mathcal{B}_\sigma D \restriction_{\widetilde{X}} = \{F \cap \widetilde{X} \mid F \in \mathcal{B}_\sigma D\}$ given by

   $$(\overline{\nu}_e)^*(E) := \inf\{\overline{\nu}_e(F) \mid E \subseteq F \text{ and } F \in \mathcal{B}_\sigma D\}$$

   and such that $(\overline{\nu}_e)^*(F \cap \widetilde{X}) \leq \overline{\nu}_e(F)$ for all $F \in \mathcal{B}_\sigma D$. By part (3) of Remark 6.8 if follows that $\nu_r(E) := (\overline{\nu}_e)^*(\widetilde{E})$ is a measure on $(X, \mathcal{B}X)$. Suppose that $G \in \mathcal{G}$. Since e is an embedding, there exists $H \in \mathcal{G}_\sigma D$ such that $G = e^{-1}H$ and

   $$\begin{aligned}
   \nu_r(G) &= (\overline{\nu}_e)^*(e[G]) && \text{by definition} \\
   &= (\overline{\nu}_e)^*(H \cap \widetilde{X}) \\
   &\leq \overline{\nu}_e(H) \\
   &= \nu_e(H) && \text{since } \overline{\nu}_e \text{ extends } \nu_e \\
   &= \nu(G)
   \end{aligned}$$

3. Notice that for all $H \in \mathcal{G}_\sigma D$ we have that $(\overline{\nu})_e(H) = \overline{\nu}(e^{-1}H) = \nu(e^{-1}H) = \nu_e(H) = \overline{\nu}_e(H)$. Since ν is finite and $\mathcal{G}_\sigma D$ is a π-system that generates $\mathcal{B}_\sigma D$, then by part (3) of Proposition 1.23 we have that $\overline{\nu}_e = (\overline{\nu})_e$.

   We claim that $(\overline{\nu}_e)^*(\widetilde{X}) = \overline{\nu}_e(D)$. Suppose that $F \in \mathcal{B}_\sigma D$ is such that $\widetilde{X} \subseteq F$. Then $\overline{\nu}_e(F) = (\overline{\nu})_e(F) = \overline{\nu}(e^{-1}F) = \overline{\nu}(X)$. This shows that $(\overline{\nu}_e)^*(\widetilde{X}) = \inf\{\overline{\nu}_e(F) \mid E \subseteq F \text{ and } F \in \mathcal{B}_\sigma D\} = \overline{\nu}(X) = \overline{\nu}_e(D)$.

   By part (2) of Proposition 1.18, it follows that $(\overline{\nu}_e)^*(F \cap \widetilde{X}) = \overline{\nu}_e(F)$ for all $F \in \mathcal{B}_\sigma D$. Now choose any $E \in \mathcal{B}X$. By part (2) of Remark 6.8, there exists $F \in \mathcal{B}_\sigma D$ such that $\widetilde{E} = F \cap \widetilde{X}$. Therefore $\nu_r(E) = (\overline{\nu}_e)^*(\widetilde{E}) = (\overline{\nu}_e)^*(F \cap \widetilde{X}) = \overline{\nu}_e(F) = \overline{\nu}(e^{-1}F) = \overline{\nu}(E)$ for all $E \in \mathcal{B}X$. ■

Remark 6.10 *Regarding part (3) of the last lemma a note of caution should be issued. Recall the definition of the valuation ν in Example 0.1. We have that $(\mathbb{N}, \mathcal{G}_\sigma\mathbb{N})$ is a T_0 space which embeds into the continuous dcpo $(\overline{\mathbb{N}}, \mathcal{G}_\sigma\overline{\mathbb{N}})$. Then $\nu_e = \overline{\nu}_e = \delta_\infty$; nevertheless ν does not extend to a Borel measure on $(\mathbb{N}, \mathcal{B}_\sigma\mathbb{N})$ and we have that $(\overline{\nu}_e)^*(\mathbb{N}) = 0$ and $\nu_r \equiv 0$.*

6.2.4 Topologies on function spaces and spaces of valuations

Suppose that $(X, \mathcal{G})$, $(Y, \mathcal{H})$ and $(Z, \mathcal{J})$ are topological spaces. Let $Y^X := \{f \mid f : X \to Y \text{ is a function}\}$. Hence $[X \to Y] = \{f \in Y^X \mid f \text{ is continuous}\}$. For a function $g : Z \times X \to Y$ we define the associate function $\widehat{g} : Z \to Y^X$ as $(\widehat{g}(z))(x) = g(z, x)$ for all $z \in Z$. A topology on $[X \to Y]$ is said to be **splitting** if for any Z the currying map

$$\begin{array}{rccc} \textbf{curry}: & [Z \times X \to Y] & \to & [Z \to [X \to Y]] \\ & g & \mapsto & \widehat{g} \end{array}$$

is well defined. That is, if $g \in [Z \times X \to Y]$, then $\widehat{g} \in [Z \to [X \to Y]]$. Conversely a topology on $[X \to Y]$ is said to be **conjoining** if the evaluation map

$$\begin{array}{rccc} \textbf{eval}: & [X \to Y] \times X & \to & Y \\ & (g, x) & \mapsto & g(x) \end{array}$$

is continuous. In fact, this is equivalent to $\widehat{g} \in [Z \to [X \to Y]]$ implies that $g \in [Z \times X \to Y]$. It follows that if a topology on $[X \to Y]$ is conjoining and splitting, then $g \in [Z \times X \to Y] \Leftrightarrow \widehat{g} \in [Z \to [X \to Y]]$. We have the following facts (see Chapter IX in [Dug66] or Section 7.7 in [Smy92b]):

- Any splitting topology is coarser (smaller) than any conjoining topology.
- Any topology on $[X \to Y]$ coarser (smaller) than a splitting topology is also splitting.
- Any topology on $[X \to Y]$ finer (larger) than a conjoining topology is also conjoining..
- There exists always a unique finest (largest) splitting topology.
- There exists at most one topology on $[X \to Y]$ which is both conjoining and splitting. In that case it is the finest of the splitting topologies and the coarsest of the conjoining topologies.

Suppose that $(X, \mathcal{G})$ and $(Y, \mathcal{H})$ are topological spaces. The **pointwise** convergence topology $\mathcal{G}_{pw}(X,Y)$ on $[X \to Y]$ is the topology that has as a subbase the sets

$$\langle x \to H \rangle = \{f \in [X \to Y] \mid f(x) \in H\}$$

with $x \in X$ and $H \in \mathcal{H}$. In fact, $\mathcal{G}_{pw}(X,Y)$ is the subspace topology from $(\Pi_{x\in X} Y, \otimes_{x\in X} \mathcal{H})$ [Dug66, XII-9-9.3].

The **compact-open** topology $\mathcal{G}_{c\_o}(X,Y)$ on $[X \to Y]$ is the topology that has as a subbase the sets

$$\langle K \to H \rangle = \{f \in [X \to Y] \mid f[K] \subseteq H\}$$

with $K \in \mathcal{K}X$ (a compact subset in X) and $H \in \mathcal{H}$.

The **Isbell** topology [Isb75b][2] $\mathcal{G}_{is}(X,Y)$ on $[X \to Y]$ is the topology with subbase

$$\langle \mathcal{U} \leftarrow H \rangle = \{f \in [X \to Y] \mid f^{-1}H \in \mathcal{U}\}$$

where $\mathcal{U}$ is a Scott-open subset of $(\mathcal{G}, \subseteq)$ (that is $\mathcal{U} \in \mathcal{G}_\sigma(\mathcal{G})$), and $H \in \mathcal{H}$. For Hausdorff spaces the Isbell and compact-open topologies agree [Isb75b, 2.5].

If Y is a mc-space, then $[X \to Y]$ is a dcpo with the pointwise order inherited from Y and we denote the Scott topology on $[X \to Y]$ by $\mathcal{G}_\sigma(X,Y)$.

Remark 6.11 *The topologies $\mathcal{G}_{pw}(X,Y), \mathcal{G}_{c\_o}(X,Y)$ and $\mathcal{G}_{is}(X,Y)$ have the same specialisation preorder. Namely $f \sqsubseteq g$ if and only if $f(x) \sqsubseteq_s g(x)$ for all $x \in X$, where $\sqsubseteq_s$ is the specialisation preorder of Y. If Y is a monotone*

[2] This paper contained some mistakes that were corrected in [Isb75a].

convergence space, then this order also agrees with the specialisation order of $\mathcal{G}_\sigma(X,Y)$.

Proof We show the first claim for the Isbell topology; the remaining cases are easier. $\boxed{\Rightarrow}$ Suppose that $f \in cl_{is}(g)$. If $f(x) \in H \in \mathcal{H}$, then $f^{-1}H \in \mathcal{N}(x)$. But it is easy to see that $\mathcal{N}(x)$ is a Scott open subset of $(\mathcal{G}, \subseteq)$, therefore $f \in \langle \mathcal{N}(x) \leftarrow H \rangle$, an Isbell open set. Then $g \in \langle \mathcal{N}(x) \leftarrow H \rangle$ from which follows that $g(x) \in H$. Since H was chosen arbitrarily, this shows that $f(x) \in cl_Y(g(x))$ for all $x \in X$. $\boxed{\Leftarrow}$ Suppose that $f(x) \in cl_Y(g(x))$ for all $x \in X$. Therefore, if $f(x) \in H \in \mathcal{H}$, then $g(x) \in H$. In other words, $f^{-1}H \subseteq g^{-1}H$ for all $H \in \mathcal{H}$. Suppose that $f \in \langle \mathcal{U} \leftarrow H \rangle$, that is $f^{-1}H \in \mathcal{U}$. Then $g^{-1}H \in \mathcal{U}$ since $\mathcal{U}$ is Scott open in $(\mathcal{G}, \subseteq)$; thus $g \in \langle \mathcal{U} \leftarrow H \rangle$. It follows that if $f \in \bigcap_{j=1}^{n} \langle \mathcal{U} \leftarrow H \rangle$, then $g \in \bigcap_{j=1}^{n} \langle \mathcal{U} \leftarrow H \rangle$. Therefore when f belongs to an Isbell basic open set so does g. That is, $f \in cl_{is}(g)$.

The second claim is trivial since the order on $[X \to Y]$ is the pointwise order inherited from Y by definition, and in a dcpo the order agrees with the specialisation order of the Scott topology. ■

Extending our previous notation for $\bullet = pw$, c_o, is, σ we set

$$[X \to Y]^\bullet := ([X \to Y], \mathcal{G}_\bullet(X,Y)).$$

If $(f_j)_{j \in J}$ is a net in $[X \to Y]$ we write $(f_j) \to_\bullet f$ if the net converges to $f \in [X \to Y]$ in the $\mathcal{G}_\bullet(X,Y)$ topology. A topological space $(X, \mathcal{G})$ is said to be **core-compact**[3] if $(\mathcal{G}, \subseteq)$ is a continuous lattice. If X is locally compact, then X is core-compact [GHK$^+$80, Proposition I-1.4]. If X is sober, then core-compactness and local compactness are equivalent [GHK$^+$80, Theorem V-5.6]. The following facts can be gathered from Section 2 in [SW85] and Section IV in [Law88]:

- $\mathcal{G}_{pw}(X,Y) \subseteq \mathcal{G}_{c\_o}(X,Y) \subseteq \mathcal{G}_{is}(X,Y)$. Moreover if Y is a monotone convergence space, then $\mathcal{G}_{is}(X,Y) \subseteq \mathcal{G}_\sigma(X,Y)$.
- $\mathcal{G}_{pw}(X,Y)$, $\mathcal{G}_{c\_o}(X,Y)$ and $\mathcal{G}_{is}(X,Y)$ are always splitting.
- If X is core-compact, then the Isbell topology $\mathcal{G}_{is}(X,Y)$ is the unique conjoining and splitting topology.
- If X is locally compact, then $\mathcal{G}_{c\_o}(X,Y) = \mathcal{G}_{is}(X,Y)$ (and is the unique conjoining and splitting topology).

[3] Also known as: semi-locally bounded, CL-spaces, quasilocally compact (not to be confused with locally quasicompact!).

From the first claim it follows that if Y is an mc-space, then $[X \to Y]^{is}$ is also a mc-space [Hec96, Section 2.3]. If $(X, \mathcal{G})$ and $(Y, \mathcal{H})$ are T_0 spaces, then the Isbell topology can also be characterised as the topology on $[X \to Y]$ which makes the function $\Omega : [X \to Y] \to [\mathcal{H}^\sigma \to \mathcal{G}^\sigma]^{pw}$ given by $\Omega(f)(H) := f^{-1}H$ a subspace embedding [Hec96]. The following proposition gives us information about the Scott topology in function spaces. Recall that for an L-domain $(D, \sqsubseteq)$, the set $\downarrow x$ is a continuous lattice for all $x \in D$. Pointed bounded complete domains are always L-domains.

Proposition 6.12 *Suppose that $(X, \mathcal{G})$ is a topological space and $(E, \sqsubseteq)$ is a dcpo. Then*

1. *If X is core-compact, then $\mathcal{G}_\sigma(X, E^\sigma)$is conjoining..*
2. *(Schwarz and Weck) If X is core-compact and E is a continuous lattice, then $\mathcal{G}_{is}(X, E^\sigma) = \mathcal{G}_\sigma(X, E^\sigma)$.*
3. *(Gierz and Keimel) If X is locally compact and E is a continuous lattice, then $\mathcal{G}_{c\_o}(X, E^\sigma) = \mathcal{G}_\sigma(X, E^\sigma)$.*
4. *(Lawson) Suppose that X is core-compact and E is a domain with least element. If $[X \to E^\sigma]$ is a stably compact domain, then $\mathcal{G}_{is}(X, E^\sigma) = \mathcal{G}_\sigma(X, E^\sigma)$.*
5. *(Ying-Ming and Ji-Hua) Suppose that E is an L-domain. Then $\mathcal{G}_{is}(X, E^\sigma) = \mathcal{G}_\sigma(X, E^\sigma)$ for all core-compact spaces X if and only if E is a bounded complete domain.*

Proof For parts (1) and (4) see Sections IV and V in [Law88]. For parts (2) and (3) see Section 2 in [SW85]. For part (5) see Theorem B in [YMJH96]. ■

In fact it is also possible to show that if X is not core-compact, then there exists no conjoining and splitting topology on $[X \to 2]$ where 2 is the Sierpinski space (defined in Section 5.3.1). A space $(X, \mathcal{G})$ is **locally finite-bottomed** if for every $x \in X$ and every $G \in \mathcal{N}(x)$, there exists a finite subset $B \subseteq_f G$ and $V \in \mathcal{N}(x)$ such that for all $H \in \mathcal{N}(B)$ we have that $V \subseteq H$. A locally finite-bottomed T_1 space is discrete.

Remark 6.13 *(Heckmann) A T_0 space $(X, \mathcal{G})$ is locally finite-bottomed if and only if for every $x \in X$ and every $G \in \mathcal{N}(x)$, there exists a finite subset $B \subseteq_f G$ such that $x \in int(\uparrow B)$.*

Proof Suppose that X is locally finite-bottomed and $x \in G \in \mathcal{G}$. Then there exists a finite $B \subseteq_f G$ and $V \in \mathcal{N}(x)$ such that for all $H \in \mathcal{N}(B)$ we have that $V \subseteq H$. We claim that $V \subseteq \uparrow B$. For if not there exists $y \in V \backslash \uparrow B$. Therefore $\downarrow y \cap \uparrow B = \emptyset$. In other words $\uparrow B \subseteq (cl(y))^{\mathsf{c}}$, which in turn implies that $(cl(y))^{\mathsf{c}} \in \mathcal{N}(B)$. By our assumption it follows that $y \in V \subseteq (cl(y))^{\mathsf{c}}$; a contradiction. The other direction is trivial by taking $V = int(\uparrow B)$ and since open sets on a T_0 space are upper. ∎

From the previous remark it is easy to verify that a locally finite-bottomed T_1 space X is discrete. Since $sat(B) = \uparrow B = B$ for any set B. If we choose any $x \in X$, then there exists a finite set B with $x \in int(B)$. Hence $int(B)$ is finite and open. Again by the T_1 axiom, for each element $z \in int(B) \backslash \{x\}$ there exists an open set $G_z \in \mathcal{N}(x)$ such that $z \notin G_z$. Therefore $\{x\} = int(B) \cap \bigcap_{z \in int(B) \backslash \{x\}} G_z$ is open. Locally finite-bottomed spaces can also be characterised as those spaces where the lattice of open sets $(\mathcal{G}, \subseteq)$ is hypercontinuous (Theorem 3.4 (1⇔3) in [SW85]). Hence, a dcpo with the Scott topology is locally finite-bottomed if and only if it is a quasicontinuous poset [GLS83, Theorem 7.2]. The following results are important. Recall from Proposition 6.4 that injective spaces are another way of referring to continuous lattices.

Proposition 6.14 *(Isbell) A non-singleton function space $[X \to Y]$ is an injective space where $\mathcal{G}_{pw}(X,Y) = \mathcal{G}_\sigma(X,Y)$ if and only if Y is an injective space and X is locally finite–bottomed.*

Proof See 2.12 in [Isb75b]. ∎

Note that if $\mathcal{G}_{pw}(X,Y) = \mathcal{G}_\sigma(X,Y)$, then $\mathcal{G}_{pw}(X,Y) = \mathcal{G}_{c\_o}(X,Y) = \mathcal{G}_{is}(X, Y) = \mathcal{G}_\sigma(X,Y)$.

Proposition 6.15 *$(X, \mathcal{G})$ is locally finite-bottomed if and only if for any topological space $(Y, \mathcal{H})$ we have that*

$$\mathcal{G}_{pw}(X,Y) = \mathcal{G}_{c\_o}(X,Y) = \mathcal{G}_{is}(X,Y).$$

If Y is a pointed bounded complete domain and $\mathcal{H} = \mathcal{G}_\sigma Y$, then the previous topologies also agree with $\mathcal{G}_\sigma(X,Y)$.

Proof The first claim follows from the equivalences (1)⇔(2)⇔(7) of Theorem 3.4 in [SW85]. Part (4) of the same theorem shows that $(X, \mathcal{G})$ is in particular core-compact, hence the second claim follows from part (5) of Proposition 6.12. ∎

Corollary 6.16 *Suppose that $(D, \sqsubseteq)$ is a continuous dcpo. Then,*

$$\mathcal{G}_{pw}(D^\sigma, [0,\infty]^{\leq}) = \mathcal{G}_{c\_o}(D^\sigma, [0,\infty]^{\leq}) = \mathcal{G}_{is}(D^\sigma, [0,\infty]^{\leq}) =$$

$$\mathcal{G}_\sigma(D^\sigma, [0,\infty]^\leq).$$

Suppose that $(X, \mathcal{H})$ *is a core-compact space and* $\mathcal{H}^\sigma = (\mathcal{H}, \mathcal{G}_\sigma(\mathcal{H}))$*; that is, the lattice* $(\mathcal{H}, \subseteq)$ *topologised with the Scott topology. Then*

$$\mathcal{G}_{pw}(\mathcal{H}^\sigma, [0,\infty]^\leq) = \mathcal{G}_{c\_o}(\mathcal{H}^\sigma, [0,\infty]^\leq) = \mathcal{G}_{is}(\mathcal{H}^\sigma, [0,\infty]^\leq) = \mathcal{G}_\sigma(\mathcal{H}^\sigma, [0,\infty]^\leq)$$

In particular this holds for $\mathcal{H} = \mathcal{G}_\sigma D$*, the Scott topology of* D*.*

Proof Notice that if D is a continuous dcpo, then by Proposition 1.10 for every $x \in D$ and $G \in \mathcal{N}(x)$, there exists $y \in G$ with $y \ll x$. Hence $x \in \twoheaduparrow y = int(\uparrow y)$ and the set $B = \{y\}$ is finite. From Remark 6.13 it follows that D^σ is locally finite-bottomed. Since $[0,\infty]^\leq$ is a continuous lattice (in particular a pointed bounded complete domain) the first claim follows from Proposition 6.15. If X is core-compact, then $(\mathcal{H}, \subseteq)$ is a continuous lattice and in particular a continuous dcpo. Thus, the second claim is obvious from the first part. For the conclusion, note that $(D, \mathcal{G}_\sigma D)$ is core-compact since it is locally compact (Proposition 1.10). ■

The last result is important since for a topological space $(X, \mathcal{H})$ we have that the space of continuous valuations $\mathbf{V}X$ is a subset of $[\mathcal{H}^\sigma \to [0,\infty]^\leq]$. In fact, $\mathbf{V}X = \{\nu \in [\mathcal{H}^\sigma \to [0,\infty]^\leq] \mid \nu \text{ is modular}\}$. For an arbitrary topological space we topologise $\mathbf{V}X$ with the subspace topology induced from $[\mathcal{H}^\sigma \to [0,\infty]^\leq]^{pw}$, and we call this topology $\mathcal{G}_{pw}(\mathbf{V}X)$. By Remark 1.4-(1) it follows that $\mathcal{G}_{pw}(\mathbf{V}X)$ has as a subbase the sets

$$\langle H \to (a,\infty]\rangle \cap \mathbf{V}X = \{\nu \in \mathbf{V}X \mid a < \nu(H)\}$$

with $H \in \mathcal{H}$ and $a \in \mathbb{R}_+$. In other words $\mathcal{G}_{pw}(\mathbf{V}X)$ is the topology of pointwise convergence on $\mathbf{V}X$.

Remark 6.17 *For a continuous dcpo* $(D, \sqsubseteq)$ *the consequences are even stronger since* $[D^\sigma \to [0,\infty]^\leq]$ *and* $\mathbf{V}D$ *can only be topologised in one natural way. Furthermore, since we know that* $(\mathbf{V}D, \sqsubseteq)$ *is also a continuous dcpo (Proposition 2.5), then* $\mathcal{G}_{pw}(\mathbf{V}D)$ *is, in fact, the Scott topology of* $(\mathbf{V}D, \sqsubseteq)$ *[Hec96, Theorem 6.9] (see also Section 4.3 in [Tix95] and Section 8.1 in [Kir93]).*

It is appropriate to mention here the work of Erker, Escardó and Keimel [EEK98] which characterises the way below relation in function spaces over continuous dcpo's. The relationship between pointwise convergence and convergence in the Scott topology is tricky and has tripped up skilled mathematicians; therefore the following facts are important and will be crucial for our discussion on weak convergence of probability measures.

Remark 6.18 ***Important facts and warning!***

Suppose that $(X, \mathcal{H})$ is a core-compact space and $(L, \sqsubseteq)$ is a continuous lattice (topologised with the Scott topology).

1. *[GHK+80, Theorem II-1.8] Suppose that $z \in L$ and $(z_j)_{j\in J}$ is a net in L. Then*

$$(z_j) \to_\sigma z \text{ if and only if } z \sqsubseteq \liminf_j z_j$$

(recall that $\liminf_j z_j \equiv \bigsqcup^{\uparrow}_{m\in J} \inf_{j\geq m} z_j$).

2. *[GHK+80, Lemma II-3.2-(i)] The Cartesian product $S := \Pi_{x\in X} L_x$ where $L_x := L$ for all $x \in X$ is a continuous lattice such that $\mathcal{G}_\sigma S = \otimes_{x\in X} \mathcal{G}_\sigma L$. That is, the Scott topology in S is the product of the Scott topologies in L.*

3. *$[X \to L] \subseteq (S, \mathcal{G}_\sigma S)$ and the subspace topology induced on $[X \to L]$ is precisely $\mathcal{G}_{pw}(X, L)$. In other words, if $(f_j)_{j\in J}$ and f are elements of $[X \to L]$, then*

$$(f_j) \to_{pw} f \text{ if and only if } f(x) \sqsubseteq \liminf_j f_j(x) \text{ for all } x \in X.$$

That is, if we fix x, then $f_j(x)$ converges to $f(x)$ in the Scott topology of L. Define $f_{pw}(x) := \liminf_j f_j(x)$ for all $x \in X$.

4. *[GHK+80, Proposition II-4.6] $[X \to L]$ is a continuous lattice, thus by part (1) we have that*

$$(f_j) \to_\sigma f \text{ if and only if } f \sqsubseteq \liminf_j f_j.$$

Set $f_\sigma(x) := (\liminf_j f_j)(x)$ for all $x \in X$.

5. *From Remark 6.11 we know that $f \in cl_{pw}(g) \Leftrightarrow f \in cl_\sigma(g) \Leftrightarrow f(x) \sqsubseteq g(x)$ for all $x \in X$ and $f, g \in [X \to L]$.*

6. *Nevertheless (the tricky bit comes here!) it does not follow that f_σ is the same function as f_{pw}. In fact $f_\sigma \in [X \to L]$ but that is not necessarily the case for f_{pw} (see example below). Since we know that convergence in the Scott topology is stronger than pointwise convergence (that is, $\mathcal{G}_{pw}(X, L) \subseteq \mathcal{G}_\sigma(X, L)$) the most that we can say is:*

$$f_\sigma \sqsubseteq f_{pw}.$$

The reader should note the analogy with pointwise convergence versus *uniform convergence for real functions.*

7. *If the net* $(f_j)_{j\in J}$ *is monotone, then* $f_\sigma(x) = (\bigsqcup_j^{\uparrow} f_j)(x) = \bigsqcup_j^{\uparrow} f_j(x) = f_{pw}(x)$.

8. *Suppose that* $(X, \mathcal{H})$ *is locally finite-bottomed (which includes: continuous lattices, continuous dcpo's, quasicontinuous posets and topologies of core-compact spaces). Hence by Proposition 6.15 if* $f \in [X \to L]$, *then*

$$(f_j) \to_\sigma f \text{ if and only if } f \sqsubseteq f_{pw};$$

and f_σ *is the greatest function in* $[X \to L]$ *below* f_{pw}. *Which means that if* $f_{pw} \in [X \to L]$, *then* $f_{pw} = f_\sigma$.

The fact that $\mathcal{G}_{pw}(X, L) \neq \mathcal{G}_\sigma(X, L)$ (in general) for a core-compact space X and a continuous lattice L was shown by J. R. Isbell (Proposition 6.14). The following example was kindly provided by Dr. R. Heckmann; we do not know if it has been published elsewhere. The author could not find an example where $f_{pw} \in [X \to L]$ but still $f_\sigma \neq f_{pw}$.

Example 6.19 *Let* $X = \overline{\mathbb{N}}^{\leq}$ *and* $L = 2^{\leq}$ *the Sierpinski space (notice that both spaces are continuous lattices). Define the following sequence of continuous functions* $f_n \in [\overline{\mathbb{N}}^{\leq} \to 2^{\leq}]$ *for all* $n \in \mathbb{N}$:

$$f_n(k) = \begin{cases} 0 & \text{if } k < n \\ 1 & \text{otherwise} \end{cases}$$

Then (f_n) *is a decreasing sequence and*

$$f_{pw}(k) = \sup_{m\in\mathbb{N}} \inf_{j\geq m} f_j(k) = \begin{cases} 0 & \text{if } k \in \mathbb{N} \\ 1 & \text{if } k = \infty \end{cases}$$

is clearly not a Scott continuous function. On the other hand $f_\sigma(k) = 0$ *for all* $k \in \overline{\mathbb{N}}$.

6.2.5 Real cones

A **real cone** or $\mathbb{R}_+$-module is a tuple $(C, +, \overline{0}, *)$ such that $+ : C \times C \to C$ is a commutative and associative operation with neutral element $\overline{0}$; and $* : \mathbb{R}_+ \times C \to C$ is an operation (scalar multiplication) satisfying the module axioms:

- $1 * \overline{x} = \overline{x}$
- $0 * \overline{x} = \overline{0}$

- $(r \cdot s) * \overline{x} = r * (s * \overline{x})$
- $r * (\overline{x} + \overline{y}) = (r * \overline{x}) + (r * \overline{y})$
- $(r + s) * \overline{x} = (r * \overline{x}) + (s * \overline{x})$

for all $r, s \in \mathbb{R}_+$ and $\overline{x}, \overline{y} \in C$. If C is a dcpo and the operations $+, *$ are Scott continuous when $\mathbb{R}_+$ is given the Scott (lower) topology, then C is called a ***d*-cone**. If C is a continuous dcpo endowed with the Scott topology, then it is called a **continuous *d*-cone**. If C is a topological space and the operations $+$ and $*$ are continuous, then $(C, +, \overline{0}, *)$ is called a **topological cone**. In other words, $+ \in [C \times C \to C]$ and $* \in [\mathbb{R}_+^{\leq} \times C \to C]$. Continuous d-cones are always topological cones (by Proposition 2.1), but for arbitrary d-cones it need not be the case that $+ \in [C \times C \to C]$. For a topological space $(X, \mathcal{G})$ the space $(\mathbf{V}X, +, 0, \cdot)$ is a (real) topological cone. Examples of continuous d-cones are: (i) $([0, \infty]^{\leq}, +, 0, \cdot)$ with $0 \cdot \infty = 0$, $\infty + \infty = \infty$ and $r \cdot \infty = \infty$ for $r > 0$; (ii) $[X \to [0, \infty]^{\leq}]^{\sigma}$ with X core-compact ; (iii) $\mathbf{V}D$ with D a continuous dcpo [Kir93].

If C are D are cones, then a function $f : C \to D$ is an homomorphism between cones if f is linear. That is,

- $f(\overline{x} + \overline{y}) = f(\overline{x}) + f(\overline{y})$ for all $\overline{x}, \overline{y} \in C$.
- $f(r * \overline{x}) = r * f(\overline{x})$ for all $r \in \mathbb{R}_+$ and $\overline{x} \in C$.

Homomorphisms between topological cones are continuous linear functions. A subset D of a real cone C is **convex** if it is closed under finite linear convex combinations. That is, if $x_i \in D$, $r_i \in \mathbb{R}_+$ for $i = 1,...,n$ and $\sum_{i=1}^{n} r_i = 1$, then $\sum_{i=1}^{n}(r_i * x_i) \in D$.

6.3 A review of the (lower) valuation integral

Next we turn our attention to the valuation integral. By valuation integral we mean the integral of a Scott continuous function $f \in [X \to [0, \infty]^{\leq}]$ with respect to a continuous valuation $\nu \in \mathbf{V}X$ on a topological space $(X, \mathcal{G})$. This idea was originally introduced by C. Jones [Jon89] and G. Plotkin [JP89] and further developed by O. Kirch [Kir93], R. Tix [Tix95], and R. Heckmann [Hec96]. Our exposition is as Heckmann's since his approach is the most general and includes the previous work. We review briefly Jones, Tix and Kirch's definitions.

- [Jon89, Section 3.9] Jones considers only finite valuations $\nu \in \mathbf{V}X$ and bounded functions $f \in [X \to \mathbb{R}_+^{\leq}]$. Suppose that f is a simple function, that is f only takes a finite number of values $\{y_1,...,y_n\}$. Then f can be rewritten as the step function $f = \sum_{i=1}^{n} y_i \mathbb{I}_{f^{-1}\{y_i\}}$, where the sets $f^{-1}\{y_i\}$ are crescents. Thus, the integral of f is defined as $\int(f,\nu) := \sum_{i=1}^{n} y_i \overline{\nu}(f^{-1}\{y_i\})$ where $\overline{\nu}$ is the extension of ν to a content on $\mathcal{A}(\mathcal{G})$. The usual linearity and monotonicity properties are shown, then $\int$ is extended to all bounded functions in $[X \to \mathbb{R}_+^{\leq}]$ via Lemma 6.3.

- [Kir93, Kapitel 4] Kirch's definition is similar but is given for any $\nu \in \mathbf{V}X$ and $f \in [X \to [0,\infty]^{\leq}]$. Again assume that f is a simple function taking a finite number of values $y_1 \leq \cdots \leq y_n$. Let $y_0 := 0$ and define $G_i := f^{-1}(y_i, \infty]$ for $k = 0,...,n-1$ noticing that $f = \sum_{i=0}^{n-1}(y_{i+1} - y_i)\mathbb{I}_{G_i}$. Then, $\int(f,\nu) = \sum_{i=0}^{n}(y_{i+1} - y_i)\overline{\nu}(G_i)$ and $\int$ extends to all functions in $[X \to [0,\infty]^{\leq}]$ via Lemma 6.3.

- [Tix95, Chapter 4] Tix also considers $\nu \in \mathbf{V}X$ and $f \in [X \to [0,\infty]^{\leq}]$. She defines an auxiliary function $f_\nu : \mathbb{R}_+ \to [0,\infty]^{\leq}$ as $f_\nu(x) = \nu(f^{-1}(x,\infty])$ for all $x \in \mathbb{R}_+$. The function f_ν is Riemann integrable since it is monotone increasing; hence $\int(f,\nu)$ is defined as $\int_0^\infty f_\nu(x)dx$.

Heckmann's approach is similar to Tix, but uses linear and universal properties of topological cones to define the integral. Notice that $\nu_f = \nu \circ f^{-1} \in \mathbf{V}[0,\infty]^{\leq}$ is the continuous valuation induced by the function f on $[0,\infty]^{\leq}$. Then it is possible to extend the identity map $Id : \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ to a continuous linear function $\overline{Id} : \mathbf{V}[0,\infty]^{\leq} \to \overline{\mathbb{R}}_+$ such that $\overline{Id}(\delta_x) = x$. Using $\overline{Id}$ the integral is defined as $\int(f,\nu) := \overline{Id}(\nu_f)$. We will not discuss here why this is well defined and possible, since the definition of the integral is basically its construction. The reader is referred to [Hec96] and [Hec95] for the full details. Heckmann defines the integral in two versions, namely Isbell and pointwise, which agree for continuous dcpo's. Here we are only interested in the Isbell version. For a topological space $(X,\mathcal{G})$, a function $f \in [X \to [0,\infty]^{\leq}]$ and a valuation $\nu \in \mathbf{V}X$, the valuation integral is denoted as $V^{+l}(f,\nu)$. The superindex "$+l$" stands for lower, since later on we define an upper version for bounded functions in $[X \to [0,\infty]^{\geq}]$ and finite valuations on $\mathbf{V}X$. For any open set $G \in \mathcal{G}$ we set $V_G^{+l}(f,\nu) = V^{+l}(f \cdot \mathbb{I}_G, \nu)$. Recall that $[X \to [0,\infty]^{\leq}]^{is}$ is the set of LSC functions $f : X \to \overline{\mathbb{R}}_+$ topologised with the Isbell topology $\mathcal{G}_{is}(X,[0,\infty]^{\leq})$ and $\mathbf{V}X$ inherits the pointwise topology from $[\mathcal{G}^\sigma \to [0,\infty]^{\leq}]^{pw}$. We have the following properties.

Proposition 6.20 *Suppose that $(X,\mathcal{G})$ is any topological space. Then the* ***lower valuation integral*** *is a function $V^{+l} : [X \to [0,\infty]^{\leq}]^{is} \times \mathbf{V}X \to$*

$[0,\infty]^{\leq}$ *satisfying:*

1. $V^{+l} : [X \to [0,\infty]^{\leq}]^{is} \times \mathbf{V}X \to [0,\infty]^{\leq}$ *is (separately) continuous in each argument.*

2. $V^{+l} : ([X \to [0,\infty]^{\leq}] \times \mathbf{V}X)^{\sigma} \to [0,\infty]^{\leq}$ *is Scott continuous. Here* $[X \to [0,\infty]^{\leq}] \times \mathbf{V}X$ *is considered as a dcpo topologised with the Scott topology.*

3. $V^{+l}(\_,\_)$ *is linear in both arguments.*

4. $V^{+l}(f,\delta_x) = f(x)$ *for all* $f \in [X \to [0,\infty]^{\leq}]$ *and* $x \in X$.

5. $V^{+l}(\mathbb{I}_G,\nu) = \nu(G)$ *for all* $G \in \mathcal{G}$ *and* $\nu \in \mathbf{V}X$.

6. *Suppose that* $(Y,\mathcal{H})$ *is a topological space,* $h \in [X \to Y]$ *and* $f \in [Y \to [0,\infty]^{\leq}]$*. Then* $V_X^{+l}(f \circ h,\nu) = V_Y^{+l}(f,\nu_h)$*, where* $\nu_h \equiv \nu \circ h^{-1}$.

7. *The topological cone* $(\mathbf{V}X,+,0,\cdot)$ *is isomorphic to the topological cone of linear continuous functionals* $[[X \to [0,\infty]^{\leq}]^{is} \overset{lin}{\to} [0,\infty]^{\leq}]^{pw}$.

8. *The topology* $\mathcal{G}_{pw}(\mathbf{V}X)$ *on* $\mathbf{V}X$ *is the weakest topology such that the function* $V^{+l}(f,\_) : \mathbf{V}X \to [0,\infty]$ *is Scott continuous (LSC) for all* $f \in [X \to [0,\infty]^{\leq}]$.

Proof See Theorems 7.1, 8.1 and 8.3 in [Hec96]. ■

The isomorphism in part (7) shows in fact that for every linear functional $\varphi : [X \to [0,\infty]^{\leq}] \to [0,\infty]^{\leq}$ there exists a continuous valuation $\nu \in \mathbf{V}X$ such that $\varphi(f) = V^{+l}(f,\nu)$ for all $f \in [X \to [0,\infty]^{\leq}]$. The reader should note that this is a purely topological version of the Riesz Representation Theorem. In this regard, Tix [Tix99a][Tix99b] has also shown Hahn-Banach type theorems for continuous d-cones.

Remark 6.21 *Suppose that* $\nu = \sum_{x\in|\nu|} r_x\delta_x$ *is any simple valuation in* $\mathbf{V}X$*. Then* ν *extends uniquely to Borel measure* $\overline{\nu}$ *on* $\mathcal{B}X$ *and for every* $f \in [X \to [0,\infty]^{\leq}]$ *by linearity we have that* $V^{+l}(f,\nu) = \sum_{x\in|\nu|} r_x f(x) = \int_X f d\overline{\nu}$ *(the Lebesgue integral of* f *with respect to* $\overline{\nu}$*).*

Proposition 6.22 *(Kirch) Suppose that* $f \in [X \to [0,\infty]^{\leq}]$ *and* ν *extends to a (Borel) measure* $\overline{\nu}$*. Then*

$$V^{+l}(f,\nu) = \int_X f d\overline{\nu}.$$

Proof By Lemma 6.3 there exists an increasing sequence of simple functions (f_n) in $[X \to [0,\infty]^{\leq}]$ with $f = \sup_{n\in\mathbb{N}} f_n$. By linearity of V^{+l} (Proposition 6.20-(3)), for every $f_n = \sum_{j=1}^{k} r_k \mathbb{I}_{G_k} \in [X \to [0,\infty]^{\leq}]$ we have that $V^{+l}(f_n, \nu) = \sum_{j=1}^{k} r_k \nu(G_k) = \int_X f_n d\overline{\nu}$. Part (2) of Proposition 6.20 implies that

$$V^{+l}(f, \nu) = \sup_{n\in\mathbb{N}} V^{+l}(f_n, \nu) = \sup_{n\in\mathbb{N}} \int_X f_n d\overline{\nu}.$$

The conclusion follows immediately from the Monotone Convergence Theorem (see Section 3.4 in [HJ94] for example). ■

Next we verify that part (7) of Proposition 6.20 still holds for any subspace of $\mathbf{V}X$ when $[X \to [0,\infty]^{\leq}]$ is replaced with $[X \to [0,1]^{\leq}]$ or the set of bounded LSC functions $\lceil X \to \mathbb{R}_{+}^{\leq} \rceil$.

Lemma 6.23 *Suppose that $(X, \mathcal{G})$ is a topological space. Then for any $S \subseteq \mathbf{V}X$ and $F = [X \to [0,1]^{\leq}]$, $\lceil X \to \mathbb{R}_{+}^{\leq} \rceil$ or $[X \to [0,\infty]^{\leq}]$ the subspace topology on S inherited from $[\mathcal{G} \to [0,\infty]^{\leq}]^{pw}$ is the weakest topology such that the function $V^{+l}(f, \_) : S \to [0,\infty]$ is Scott continuous (LSC) for every $f \in F$.*

Proof We check the statement for $F = [X \to [0,1]^{\leq}]$; the remaining cases are similar. Let $\mathcal{G}_{pw}(S)$ denote the subspace topology on S induced by the topology $\mathcal{G}_{pw}(\mathcal{G},[0,\infty]^{\leq})$. For simplicity for every $f \in [X \to [0,\infty]^{\leq}]$ we define $\phi_f : \mathbf{V}X \to [0,\infty]^{\leq}$ as $\phi_f(\nu) := V^{+l}(f, \nu)$. By Remark 1.4-(1) we know that $\mathcal{G}_{pw}(S)$ is generated by the paving $\{S \cap \langle G \to (a,\infty]\rangle \mid G \in \mathcal{G}$ and $a \in \mathbb{R}_+\}$. Now observe that:

$$\begin{aligned} S \cap \langle G \to (a,\infty]\rangle &= \{\nu \in S \mid a < \nu(G)\} \\ &= \{\nu \in S \mid a < V^{+l}(\mathbb{I}_G, \nu)\} \\ &= \{\nu \in S \mid a < \phi_{\mathbb{I}_G}(\nu)\} \\ &= S \cap \phi_{\mathbb{I}_G}^{-1}(a,\infty] \end{aligned}$$

with $\mathbb{I}_G \in [X \to [0,1]^{\leq}]$. This shows that the weakest topology on S which makes all the functions in the set $\{\phi_f \mid f \in [X \to [0,1]^{\leq}]\}$ continuous, is finer than $\mathcal{G}_{pw}(S)$.

We claim that $\mathcal{G}_{pw}(S) = \mathcal{G}_{pw}(\mathbf{V}X) \restriction_S$. Since $\mathcal{G}_{pw}(\mathbf{V}X) = \mathcal{G}_{pw}(\mathcal{G},[0,\infty]^{\leq}) \restriction_{\mathbf{V}X}$ and $S \subseteq \mathbf{V}X$, then by part (2) of Remark 1.4 we have that

$$\begin{aligned} \mathcal{G}_{pw}(S) &\equiv \mathcal{G}_{pw}(\mathcal{G},[0,\infty]^{\leq}) \restriction_S \\ &= (\mathcal{G}_{pw}(\mathcal{G},[0,\infty]^{\leq}) \restriction_{\mathbf{V}X}) \restriction_S \\ &\equiv \mathcal{G}_{pw}(\mathbf{V}X) \restriction_S \end{aligned}$$

Now take any $f \in [X \to [0,1]^{\leq}]$, since $[X \to [0,1]^{\leq}] \subseteq [X \to [0,\infty]^{\leq}]$, then by part (7) of Proposition 6.20 the function $\phi_f : \mathbf{V}X \to [0,\infty]^{\leq}$ is continuous. Hence for every $a \in \mathbb{R}_+$ we have that $\phi_f^{-1}(a,\infty] \in \mathcal{G}_{pw}(\mathbf{V}X)$. Therefore $S \cap \phi_f^{-1}(a,\infty] \in \mathcal{G}_{pw}(\mathbf{V}X) \restriction_S = \mathcal{G}_{pw}(S)$. Which shows that the weakest topology on S which makes all the functions in the set $\{\phi_f \mid f \in [X \to [0,1]^{\leq}]\}$ continuous, is coarser than $\mathcal{G}_{pw}(S)$ and the conclusion follows. ■

Corollary 6.24 *The topology $\mathcal{G}_{pw}(\mathbf{P}^1X)$ in $\mathbf{P}^1X$ is the weakest topology such that the function $V^{+l}(f,\_) : \mathbf{P}^1X \to [0,\infty]^{\leq}$ is Scott continuous (USC) for all $f \in [X \to [0,\infty]^{\leq}]$.*

The reader should note that the previous lemma characterises the topologies on the following subspaces of $\mathbf{V}X$ as weak topologies for the lower valuation integral.

- $\mathbf{P}X = \{\nu \in \mathbf{V}X \mid \nu(X) \leq 1\}$
- $\mathbf{P}^rX := \{\nu \in \mathbf{V}X \mid \nu(X) = r\}$ defined for $r \in (0,\infty)$ and called the r-**normalised** valuation space of X.
- $\mathbf{V}_+X := \{\nu \in \mathbf{V}X \mid \nu(X) < \infty\}$ is called the **finite** valuation space of X.

While $\mathbf{P}X$ and $\mathbf{P}^rX$ are closed under directed suprema (i.e., they are sub-dcpo's of $\mathbf{V}X$), $\mathbf{V}_+X$ is not; which makes it cumbersome to work with. Set $\mathbf{L}X := [[X \to [0,\infty]^{\leq}]^{is} \overset{lin}{\to} [0,\infty]^{\leq}]$, that is the topological cone of linear continuous functionals from $[X \to [0,\infty]^{\leq}]^{is}$ to $[0,\infty]^{\leq}$.

The following remark simplifies considerably the proofs of the forthcoming results; it was brought to the author's attention by Dr. R. Heckmann.

Remark 6.25 *Suppose that $(X,\mathcal{G})$ and $(Y,\mathcal{H})$ are monotone convergence spaces. If $f : X \to Y$ is continuous, then $f : X \to Y$ is Scott continuous. That is, $[X \to Y] \subseteq [X^\sigma \to Y^\sigma]$.*

Proof Follows from Lemma 1 in [Ers99]. ■

Lemma 6.26 *Suppose that $(X,\mathcal{G})$ is a topological space. Then $\mathbf{P}^1X$ (as a convex subset of $(\mathbf{V}X,+,0,\cdot)$) is isomorphic to the convex subsets of linear continuous functionals*

1. $\{\gamma \in \mathbf{L}X \mid \gamma(1) = 1\}$.
2. $\{\gamma \in [[X \to [0,2]^{\leq}]^{is} \overset{lin}{\to} [0,2]^{\leq}]^{pw} \mid \gamma(1) = 1\}$.

Proof

$\boxed{1}$ It suffices to check that the restriction of the isomorphism between $\mathbf{L}X$ and $\mathbf{V}X$ gives us the desired isomorphism. If $\nu \in \mathbf{P}^1X$, then the functional $\gamma_\nu(f) := V^{+l}(f,\nu)$ is an element of $\mathbf{L}X$ and clearly $\gamma_\nu(1) := V^{+l}(1,\nu) = \nu(X) = 1$. Conversely if $\gamma \in \mathbf{L}X$ with $\gamma(1) = 1$, then by part (7) of Proposition 6.20, there exists $\nu_\gamma \in \mathbf{V}X$ with $\gamma(f) = V^{+l}(f,\nu_\gamma)$ for all $f \in [X \to [0,\infty]^{\leq}]$. Clearly $\nu_\gamma(X) = V^{+l}(1,\nu_\gamma) = \gamma(1) = 1$, hence $\nu_\gamma \in \mathbf{P}^1X$.

$\boxed{2}$ Suppose that $\nu \in \mathbf{P}^1X$. Then the functional γ_ν constructed in part (1) is linear, and for every $f \in [X \to [0,2]^{\leq}]$ we have that $\gamma_\nu(f) = V^{+l}(f,\nu) \leq V^{+l}(2,\nu) = 2\nu(X) = 2$. Conversely suppose that $\gamma \in [[X \to [0,2]^{\leq}]^{is} \xrightarrow{lin} [0,2]^{\leq}]$ with $\gamma(1) = 1$. Then γ is Scott continuous by Remark 6.25. In the following $G_j \in \mathcal{G}$. Then,

- $\gamma(\mathbb{I}_\emptyset) = \gamma(0) = 0$.
- $\gamma(\mathbb{I}_X) = \gamma(1) = 1$.
- If $G_1 \subseteq G_2$, then $\mathbb{I}_{G_1} \leq \mathbb{I}_{G_2}$. Since γ is Scott continuous, $\gamma(\mathbb{I}_{G_1}) \leq \gamma(\mathbb{I}_{G_2})$.
- $\gamma(\mathbb{I}_{G_1\cup G_2}) + \gamma(\mathbb{I}_{G_1\cap G_2}) = \gamma(\mathbb{I}_{G_1\setminus G_2} + \mathbb{I}_{G_1\cap G_2} + \mathbb{I}_{G_2\setminus G_1}) + \gamma(\mathbb{I}_{G_1\cap G_2}) = \gamma(\mathbb{I}_{G_1\setminus G_2} + \mathbb{I}_{G_1\cap G_2}) + \gamma(\mathbb{I}_{G_1\cap G_2} + \mathbb{I}_{G_2\setminus G_1}) = \gamma(\mathbb{I}_{G_1}) + \gamma(\mathbb{I}_{G_2})$.
- If $G = \bigcup^{\uparrow}_{j\in J} G_j$, then $\mathbb{I}_G = \bigsqcup^{\uparrow}_{j\in J} \mathbb{I}_{G_j}$. Since γ is Scott continuous, $\gamma(\mathbb{I}_G) = \sup_{j\in J} \gamma(\mathbb{I}_{G_j})$.

Hence if we define $\nu_\gamma(G) = \gamma(\mathbb{I}_G)$ for all $G \in \mathcal{G}$, then clearly $\nu_\gamma \in \mathbf{P}^1X$. Since every $f \in [X \to [0,2]^{\leq}]$ is the limit of an increasing sequence of simple functions (Lemma 6.3) and γ is linear, it is not difficult to see that $\gamma(f) = V^{+l}(f,\nu_\gamma)$. The rest of the argument is as in part (1). ■

Similarly it is possible to show that $\mathbf{P}X \cong \{\varphi \in \mathbf{L}X \mid \varphi(1) \leq 1\}$, $\mathbf{P}^rX \cong \{\varphi \in \mathbf{L}X \mid \varphi(1) = r\}$ and $\mathbf{V}_+X \cong \{\varphi \in \mathbf{L}X \mid \varphi(1) < \infty\}$.

6.4 A review of the Riemann-Edalat integral

In [Eda95d] the upper space (see Section 5.3.2) of a locally compact Hausdorff space was introduced as a computational model for locally compact Hausdorff spaces. This opened a new alley for doing measure theory via domain theory. It was shown that for a locally compact second countable Hausdorff space $(X,\mathcal{G})$, the set of probability measures[4] $\mathfrak{M}^1(X)$ on X is isomorphic to the set $\{\nu \in \mathbf{P}^1(\mathbf{U}X) \mid \overline{\nu}^*(\max(\mathbf{U}X)) = 1\}$; that is, the

[4]Recall that on a locally compact second countable Hausdorff space $\mathfrak{M}^1(X) = \mathfrak{M}^1(X;t) = \mathfrak{M}^1(X;\tau) = \mathfrak{M}^1(X;r)$ by Corollary 1.27 and Proposition 1.28.

set of normalised continuous valuations on $\mathbf{U}X$ supported on the maximum elements. If $e : X \to \max(\mathbf{U}X)$ is the subspace embedding, then the isomorphism is given by $\mu \mapsto \mu_e = \mu \circ e^{-1}$ for all $\mu \in \mathfrak{M}^1(X)$ and $\nu \mapsto \nu_r$ for all $\nu \in \mathbf{P}^1(\mathbf{U}X)$ such that $\overline{\nu}^*(\max(\mathbf{U}X)) = 1$, where ν_r is the measure constructed in Corollary 6.9. Edalat [Eda95b] developed these ideas further to show that for any $\mu \in \mathfrak{M}^1(X)$ on a compact metric space (X, d), the directed set of simple valuations which approximates μ_e (given by Proposition 2.5) could be used to give partial upper and lower approximations to the value of the integral $\int f d\mu$ of any (bounded) $f : X \to \mathbb{R}$. These partial approximations determine upper and lower integrals which agree with the value of the Lebesgue integral of f if $\mu(D_f) = 0$; giving thus a generalisation of the Riemann integral for compact metric spaces. This work was later extended by Edalat and Negri [EN98] to locally compact Hausdorff spaces. Recently, Howroyd [How00] has given a further extension to all pointed S-domain representations of a Hausdorff space $(X, \mathcal{G})$. We review Howroyd's construction of the R-integral.

Consider the following setting. Let $(X, \mathcal{G})$ be a Hausdorff topological space, $\mathcal{B}X$ its Borel σ-algebra and $(D, \sqsubseteq, e, \perp)$ be a pointed S-domain representation for X (Definition 5.1). Suppose that $\mu \in \mathfrak{M}^1(X, \tau)$ (that is, μ is a τ-smooth probability measure on $\mathcal{B}X$) and μ_e is the normalised continuous valuation in $\mathbf{P}^1D$ given by $\mu_e(G) = \mu(e^{-1}G)$ for all $G \in \mathcal{G}_\sigma D$ (recall the statement of Corollary 6.9). For any $\nu \in \mathbf{P}^1D$ let $\overline{\nu} \in \mathfrak{M}^1(D, \tau)$ denote the unique extension of ν to a τ-smooth probability measure on $\mathcal{B}D^\sigma$ (given by Corollary 2.22). Recall from Remark 6.21 that if $\nu = \sum_{x \in |\nu|} r_x \delta_x$ is any simple valuation in $\mathbf{P}^1D$, then for $g : D \to \mathbb{R}$ we have that $\int g d\overline{\nu} = \sum_{x \in |\nu|} r_x g(x)$. Define

$$S(\mu_e) := \{\nu \in \mathbf{P}^1D \mid \nu \ll \mu_e \text{ and } \nu \text{ simple}\}$$

For every bounded $f : X \to \mathbb{R}$ let the functions f_*, $f_{(*)}$, $f_{[*]} : D \to [-M_f, M_f]$ be defined as:

$$\begin{aligned}
f_*(d) &= \inf\{f(x) \mid x \in e^{-1}{\uparrow}d\} \\
&= \inf f[e^{-1}{\uparrow}d] \\
f_{(*)}(d) &= \inf\{f(x) \mid x \in e^{-1}{\Uparrow}d\} \\
&= \inf f[e^{-1}{\Uparrow}d] \\
f_{[*]}(d) &= \sup\nolimits_{c \ll d} f_{(*)}(c) \\
&= \sup\{\inf f[e^{-1}{\Uparrow}c] \mid c \ll d \text{ and } c \in D\}.
\end{aligned}$$

It is not difficult to see that $f_{[*]} \leq f_* \leq f_{(*)}$ and by Lemma 26 in [How00] the **lower R-integral** $R^l(f, \mu)$ can be defined in any of the following three ways:

$$
\begin{aligned}
R^l(f,\mu) &= \sup_{\nu\in S(\mu_e)} \int f_{[*]} d\overline{\nu} \\
&= \sup_{\nu\in S(\mu_e)} \int f_{*} d\overline{\nu} \\
&= \sup_{\nu\in S(\mu_e)} \int f_{(*)} d\overline{\nu}
\end{aligned}
$$

Similarly let $f^{(*)}$, f^*, $f^{[*]} : D \to [-M_f, M_f]$ be defined as

$$
\begin{aligned}
f^*(d) &= \sup f[e^{-1}{\uparrow}d] \\
f^{(*)}(d) &= \sup f[e^{-1}{\Uparrow}d] \\
f^{[*]}(d) &= \inf_{c\ll d} f^{(*)}(c) \\
&= \inf\{\sup f[e^{-1}{\Uparrow}c] \mid c \ll d \text{ and } c \in D\}
\end{aligned}
$$

Then $f^{(*)} \le f^* \le f^{[*]}$ and the **upper R-integral** $R^u(f,\mu)$ is defined as

$$
\begin{aligned}
R^u(f,\mu) &= \inf_{\nu\in S(\mu_e)} \int f^{(*)} d\overline{\nu} \\
&= \inf_{\nu\in S(\mu_e)} \int f^{*} d\overline{\nu} \\
&= \inf_{\nu\in S(\mu_e)} \int f^{[*]} d\overline{\nu}
\end{aligned}
$$

We say that f is **R-integrable** with respect to μ if $R^l(f,\mu) = R^u(f,\mu)$ and we denote this common value as $R(f,\mu)$. We set $\mathcal{R}(\mu)$ to be the set of R-integrable functions with respect to μ. We have the following properties:

Proposition 6.27 *Suppose that $(X,\mathcal{G})$ is a Hausdorff topological space, $(D,\sqsubseteq)$ is a continuous dcpo with least element $\perp$, $e : X \to D^\sigma$ is a subspace embedding. Suppose that $f : X \to \mathbb{R}_+$ is a bounded function and $\mu \in \mathfrak{M}^1(X,\tau)$, then*

1. *If $\nu_1, \nu_2 \in S(\mu_e)$ and $\nu_1 \sqsubseteq \nu_2$, then $\int f_* d\overline{\nu}_1 \le \int f_* d\overline{\nu}_2$ and $\int f^* d\overline{\nu}_2 \le \int f^* d\overline{\nu}_1$.*
2. *If $\nu_1, \nu_2 \in S(\mu_e)$, then $\int f_* d\overline{\nu}_1 \le \int f^* d\overline{\nu}_2$.*
3. *$R^l(f,\mu) \le R^u(f,\mu)$.*
4. *f is R-integrable if and only if $\mu(D_f) = 0$. In which case $R(f,\mu) = \int f d\mu$.*
5. *$R^l(\_,\_)$ is superlinear in the first argument and $\mathcal{R}(\mu)$ is a real vector space.*

Proof See Sections 3.2 to 4.3 in [How00]. ■

Next Howroyd [How00, Section 4.4] proceeds to obtain the Bourbaki extension of R and shows that for completely regular Hausdorff spaces it recovers the Lebesgue integral. Furthermore in Section 7.4 *loc cit* Howroyd introduces yet another definition of integral which with an appropriate extension recovers the Lebesgue integral.

6.5 Upper and real-valued valuation integrals

Next we define an upper valuation integral. Throughout the section $(X, \mathcal{G})$ denotes a topological space.

Definition 6.28 *Suppose that $f \in [X \to \mathbb{R}^{\geq}_{\mp}]$ is bounded and $\nu \in \mathbf{V}_+X$. Then the function $x \mapsto M_f - f(x)$ belongs to $[X \to [0, M_f]^{\leq}]$. Hence we define the* ***upper valuation integral*** *$V^{+u}(f, \nu)$ as:*

$$V^{+u}(f, \nu) := M_f \cdot \nu(X) - V^{+l}(M_f - f, \nu)$$

Notice that $V^{+l}(M_f - f, \nu) \leq V^{+l}(M_f, \nu) = M_f \cdot \nu(X)$, therefore $V^{+u}(f, \nu)$ is finite. If there exists another constant N_f such that f is bounded by N_f, then $N_f \leq M_f$ or $M_f \leq N_f$. Without loss of generality assume that $N_f = M_f + c$ with $c \geq 0$. Then since $(M_f - f) \geq 0$ by Proposition 6.20-(3) it follows that

$$\begin{aligned} N_f \cdot \nu(X) - V^{+l}(N_f - f, \nu) &= (M_f + c) \cdot \nu(X) - V^{+l}(M_f + c - f, \nu) \\ &= (M_f + c) \cdot \nu(X) - [V^{+l}(M_f - f, \nu) \\ &\quad + V^{+l}(c, \nu)] \\ &= M_f \cdot \nu(X) - V^{+l}(M_f - f, \nu). \end{aligned}$$

This shows that V^{+u} is well defined and does not depend on the value of the bounding constant M_f. We verify which of the properties in Proposition 6.20 still hold for the upper integral through a series of lemmata. Having restricted ourselves to bounded functions and finite valuations introduces new complications. Hence, the theory cannot be presented in a neat and elegant way and we have to deal with some special cases.

Lemma 6.29 *Suppose that $(X, \mathcal{G})$ is a topological space. Then $V^{+u} : \lceil X \to \mathbb{R}^{\geq}_{\mp}\rceil \times \mathbf{V}_+X \to \mathbb{R}^{\geq}_{\mp}$ satisfies the following properties.*

1. *$V^{+u}(\_, \_)$ is linear in both arguments.*
2. *$V^{+u}(g, \delta_x) = g(x)$ for all $g \in \lceil X \to \mathbb{R}^{\geq}_{\mp}\rceil$ and $x \in X$.*
3. *$V^{+u}(\mathbb{I}_F, \nu) = \nu(X) - \nu(F^{\mathrm{c}})$ for all $F \in \mathcal{F}$ and $\nu \in \mathbf{P}X$.*
4. *Suppose that $(Y, \mathcal{H})$ is a topological space, $h \in [X \to Y]$ and $f \in \lceil Y \to \mathbb{R}^{\geq}_{\mp}\rceil$. Then $V^{+u}_X(f \circ h, \nu) = V^{+u}_Y(f, \nu_h)$, where $\nu_h \equiv \nu \circ h^{-1}$.*

Proof Parts (2) and (3) are immediate from the definitions.

$\boxed{1}$ Suppose that $r, s \in \mathbb{R}_+$, $\nu, \mu \in \mathbf{V}_+X$ and $g_1, g_2, g \in \lceil X \to \mathbb{R}^{\geq}_{+} \rceil$. Then $rg_1 + sg_2 \leq rM_{g_1} + sM_{g_2}$ and by part (3) of Proposition 6.20 it follows that,

$$\begin{aligned} V^{+u}(rg_1 + sg_2, \nu) &= rM_{g_1} + sM_{g_2} - V^{+l}(rM_{g_1} + sM_{g_2} - (rg_1 + sg_2), \nu) \\ &= rM_{g_1} + sM_{g_2} - V^{+l}(r(M_{g_1} - g_1) + s(M_{g_2} - g_2), \nu) \\ &= rV^{+u}(g_1, \nu) + sV^{+u}(g_2, \nu). \end{aligned}$$

Similarly it is possible to show that $V^{+u}(g, r\nu + s\mu) = rV^{+u}(g, \nu) + sV^{+u}(g, \mu)$.

$\boxed{4}$ If M_g if an upper bound for g, then so it is for $g \circ h$. Hence, by part (6) of Proposition 6.20,

$$\begin{aligned} V_D^{+u}(g \circ h, \nu) &= M_g - V_D^{+l}(M_g - g \circ h, \nu) \\ &= M_g - V_D^{+l}((M_g - g) \circ h, \nu) \\ &= M_g - V_E^{+l}(M_g - g, \nu_h) \\ &= V_E^{+u}(g, \nu_h). \end{aligned}$$

■

Before we state the convergence properties it is important to notice the following:

Remark 6.30 *If we want $V^{+u}(f, \_)$ to be monotone in $\mathbb{R}^{\geq}_{+}$ for every $f \in \lceil X \to \mathbb{R}^{\geq}_{+} \rceil$, in particular, this needs to hold for the constant function $1 = \mathbb{I}_X$. Hence if $\nu, \mu \in \mathbf{V}_+X$ are such that $\nu \sqsubseteq \mu$, then we want $\nu(X) = V^{+u}(\mathbb{I}_X, \nu) \geq V^{+u}(\mathbb{I}_X, \mu) = \mu(X)$ to hold. But we already know that $\nu(X) \leq \mu(X)$. Therefore $V^{+u}(f, \_)$ is a monotone function for all $f \in \lceil X \to \mathbb{R}^{\geq}_{+} \rceil$ only if we restrict its domain to $\mathbf{P}^rX$ for some $r \in (0, \infty)$. Notice that for $\nu, \mu \in \mathbf{P}^rX$ we have that $\nu \sqsubseteq \mu$ if and only if $\overline{\mu}(F) \leq \overline{\nu}(F)$ for all $F \in \mathcal{F}$, where $\overline{\nu}$ and $\overline{\mu}$ are the extensions of ν and μ to contents on $\mathcal{A}(\mathcal{G})$ given by Proposition 1.23.*

For simplicity we formulate the convergence results for $\mathbf{P}^1X$; the reader should note that everything that is said for $\mathbf{P}^1X$ it is also true for $\mathbf{P}^rX$ with $r \in (0, \infty)$. We will use the results on net convergence in Proposition 1.9.

Lemma 6.31 ***Continuity in the first argument.*** *Fix $\nu \in \mathbf{V}_+X$ and suppose that $M < \infty$, then*

$$V^{+u}(\_, \nu) : [X \to [0, M]^{\geq}]^{is} \to \mathbb{R}^{\geq}_{+} \text{ is continuous.}$$

Proof Recall that the Isbell subbasic open sets in $\mathcal{G}_{is}(X, [0, M]^{\geq})$ are of the form $\langle \mathcal{U} \leftarrow [0, b) \rangle = \{f \in [X \to [0, M]^{\geq}] \mid f^{-1}[0, b) \in \mathcal{U}\}$ where $\mathcal{U}$ is a Scott-open subset of $(\mathcal{G}, \subseteq)$ and $b \in [0, M]$. Notice that $f^{-1}[0, b) = \{x \mid 0 \le f(x) < b\} = \{x \mid M - b < (M - f)(x) \le M\} = (M - f)^{-1}(M - b, M]$. Therefore

$$f \in \langle \mathcal{U} \leftarrow [0, b) \rangle \text{ if and only if } (M - f) \in \langle \mathcal{U} \leftarrow (M - b, M] \rangle.$$

This shows that the function

$$\begin{array}{rcl} \Psi_1 : [X \to [0, M]^{\ge}]^{is} & \to & [X \to [0, M]^{\le}]^{is} \\ f & \mapsto & M - f \end{array}$$

is continuous. By part (1) of Proposition 6.20 we have that

$$V^{+u}(\_, \nu) : [X \to [0, M]^{\le}]^{is} \to [0, M \cdot \nu(X)]^{\le}$$

is continuous. Finally, it is obvious that the following function is also continuous

$$\begin{array}{rcl} \Psi_2 : [0, M \cdot \nu(X)]^{\le} & \to & [0, M \cdot \nu(X)]^{\ge} \\ r & \mapsto & M \cdot \nu(X) - r \end{array}$$

Since $V^{+u}(\_, \nu) = \Psi_2 \circ V^{+u}(\_, \nu) \circ \Psi_1$, the conclusion follows. ■

Lemma 6.32 ***Continuity in the second argument.*** *Fix $f \in \lceil X \to \mathbb{R}_{+}^{\ge} \rceil$, then*

$$V^{+u}(f, \_) : \mathbf{P}^1 X \to \mathbb{R}_{+}^{\ge} \textit{ is continuous.}$$

In fact, if $(\nu_j)_{j \in J}$ is a net in $\mathbf{V}_+ X$ such that $(\nu_j) \to_{pw} \nu \in \mathbf{V}_+ X$ and $\limsup_j \nu_j(X) \le \nu(X)$, then $V^{+u}(f, \nu_j) \to V^{+u}(f, \nu)$ in the Scott topology of $\mathbb{R}_{+}^{\ge}$.

Proof Suppose that $(\nu_j)_{j \in J}$ is a net in $\mathbf{V}_+ X$ such that $(\nu_j) \to_{pw} \nu \in \mathbf{V}_+ X$ and $\limsup_j \nu_j(X) \le \nu(X)$. Then by part (1) of Proposition 6.20 it follows that $V^{+l}(M_f - f, \nu_j) \to V^{+l}(M_f - f, \nu)$ in the Scott topology of $\mathbb{R}_{+}^{\le}$. By part (1) of Remark 6.18 this means that $V^{+l}(M_f - f, \nu) \le \liminf_j V^{+l}(M_f - f, \nu_j)$. In other words $\limsup_j -V^{+l}(M_f - f, \nu_j) \le -V^{+l}(M_f - f, \nu)$. Since $\limsup_j \nu_j(X) \le \nu(X)$, then

$$\begin{aligned} \limsup_j V^{+u}(f, \nu_j) &\equiv \limsup_j (M_f \cdot \nu_j(X) - V^{+l}(M_f - f, \nu_j)) \\ &\le M_f \cdot \nu(X) - V^{+l}(M_f - f, \nu) \\ &= V^{+u}(f, \nu) \end{aligned}$$

Thus, $V^{+u}(f, \nu_j) \to V^{+u}(f, \nu)$ in the Scott topology of $\mathbb{R}_{+}^{\ge}$. If $(\nu_j)_{j \in J}$ and ν are elements of $\mathbf{P}^1 X$, then $\limsup_j \nu_j(X) = 1 = \nu(X)$ and the conclusion follows. ■

Notice that if $(\nu_j) \to_{pw} \nu \in \mathbf{V}_+X$, then $\nu(X) \leq \liminf_j \nu(X)$. Therefore,

$$\limsup_j \nu_j(X) \leq \nu(X) \text{ if and only if } \nu(X) = \lim_j \nu_j(X).$$

Lemma 6.33 ***Scott continuity in the first argument.*** *Fix $\nu \in \mathbf{V}_+X$, then*

$$V^{+u}(\_,\nu) : \lceil X \to \mathbb{R}^{\geq}_{\mp}\rceil \to \mathbb{R}^{\geq}_{\mp} \textit{ is Scott continuous.}$$

Proof Suppose that $\langle f_j \rangle_{j\in J}$ is a monotone net in $\lceil X \to \mathbb{R}^{\geq}_{\mp}\rceil$. Fix $k \in J$. Since $\langle f_j \rangle_{j\in J}$ is monotone, for each $j \in J$ there exists $j_k \in J$ with f_j, $f_k \sqsubseteq f_{j_k}$. Define $g_j := f_{j_k}$ then $\langle g_j \rangle_{j\in J}$ is a monotone net in $[X \to [0, M]^{\geq}]$ where M is an upper bound for f_k. Clearly $f := \bigsqcup^{\uparrow}_{j\in J} f_j = \bigsqcup^{\uparrow}_{j\in J} g_j$, since the nets $\langle f_j \rangle_{j\in J}$ and $\langle g_j \rangle_{j\in J}$ are cofinal. By Lemma 6.31 we know that the function

$$V^{+u}(\_,\nu) : [X \to [0, M]^{\geq}]^{is} \to \mathbb{R}^{\geq}_{\mp}$$

is continuous. By Remark 6.25 it follows that $V^{+u}(\_,\nu)$ is Scott continuous. Hence, $V^{+u}(f,\nu) = \bigsqcup^{\uparrow}_j V^{+u}(g_j,\nu) = \bigsqcup^{\uparrow}_j V^{+u}(f_j,\nu)$ and the conclusion follows. ■

Lemma 6.34 ***Scott continuity in the second argument.*** *Fix $f \in \lceil X \to \mathbb{R}^{\geq}_{\mp}\rceil$, then*

$$V^{+u}(f,\_) : \mathbf{P}^1X \to \mathbb{R}^{\geq}_{\mp} \textit{ is Scott continuous.}$$

Proof By Lemma 6.32 we have that $V^{+u}(f,\_) : \mathbf{P}^1X \to \mathbb{R}^{\geq}_{\mp}$ is continuous. The conclusion follows from Remark 6.25. ■

If $\langle \nu_j \rangle_{j\in J}$ is a monotone net in $\mathbf{V}_+X$ such that $\nu := \bigsqcup^{\uparrow}_{j\in J} \nu_j \in \mathbf{V}_+X$, then $\nu(X) = \sup_j \nu_j(X) = \lim_j \nu_j(X)$. Hence by Lemma 6.32 we have that $V^{+u}(f,\nu_j) \to V^{+u}(f,\nu)$ in the Scott topology of $\mathbb{R}^{\geq}_{\mp}$. Nevertheless the reader should not be confused into believing that $V^{+u}(f,\_) : \mathbf{V}_+X \to \mathbb{R}^{\geq}_{\mp}$ is Scott continuous since, as we noted in Remark 6.30, $V^{+u}(f,\_)$ will only be monotone when its domain is restricted to $\mathbf{P}^rX$ for some r.

Lemma 6.35 ***Pointwise topology is the weak topology.*** *The topology $\mathcal{G}_{pw}(\mathbf{P}^1X)$ in $\mathbf{P}^1X$ is the weakest topology such that the function $V^{+u}(g,\_) : \mathbf{P}^1X \to \mathbb{R}^{\geq}_{\mp}$ is Scott continuous (USC) for all $g \in \lceil X \to \mathbb{R}^{\geq}_{\mp}\rceil$.*

The proof is very similar to the proof of Lemma 6.23; we present it for completeness.

Proof Recall from the proof of Lemma 6.23 that for every $f \in [X \to [0,\infty]^{\leq}]$ the function $\phi_f : \mathbf{V}X \to [0,\infty]^{\leq}$ is defined as $\phi_f(\nu) = V^{+l}(f,\nu)$. Similarly, we define $\varphi_g : \mathbf{P}^1 X \to \mathbb{R}_+$ as $\varphi_g(\nu) := V^{+u}(g,\nu)$ for all $g \in \lceil X \to \mathbb{R}_+^{\geq} \rceil$. By Remark 1.4-(1) we know that $\mathcal{G}_{pw}(\mathbf{P}^1 X)$ is generated by the paving $\{\mathbf{P}^1 X \cap \langle G \to (a,1] \rangle \mid G \in \mathcal{G} \text{ and } a \in [0,1]\}$. Now observe that

$$\begin{aligned}
\mathbf{P}^1 X \cap \langle G \to (a,1] \rangle &= \{\nu \in \mathbf{P}^1 X \mid a < \nu(G) \leq 1\} \\
&= \{\nu \in \mathbf{P}^1 X \mid 0 \leq \nu(X) - \nu(G) < 1 - a\} \\
&\qquad\qquad\qquad\qquad \text{since } \nu(X) = 1 \\
&= \{\nu \in \mathbf{P}^1 X \mid 0 \leq V^{+u}(\mathbb{I}_{G^c}, \nu) < 1 - a\} \\
&= \varphi^{-1}_{\mathbb{I}_{G^c}}[0, 1-a)
\end{aligned}$$

with $\mathbb{I}_{G^c} \in \lceil X \to \mathbb{R}_+^{\geq} \rceil$. This shows that the weakest topology $\mathcal{W}$ on $\mathbf{P}^1 X$ which makes all the functions in the set $\{\varphi_g \mid g \in \lceil X \to \mathbb{R}_+^{\geq} \rceil\}$ continuous, is finer than $\mathcal{G}_{pw}(\mathbf{P}^1 X)$. On the other hand, by Lemma 6.32 we know that all the functions in the set $\{\varphi_g \mid g \in \lceil X \to \mathbb{R}_+^{\geq} \rceil\}$ are continuous. Hence $\mathcal{W} \subseteq \mathcal{G}_{pw}(\mathbf{P}^1 X)$ and the conclusion follows. ■

Lemma 6.36 *Suppose that $(X, \mathcal{G})$ is a topological space. Then, $\mathbf{P}^1 X$ (as a convex subset of $(\mathbf{V}X, +, 0, \cdot)$) is isomorphic to $\{\beta \in [[X \to [0,2]^{\geq}]^{is} \overset{lin}{\to} [0,2]^{\geq}]^{pw} \mid \beta(1) = 1\}$.*

Proof By part (2) of Lemma 6.26 it suffices to show that $L_1 := \{\gamma \in [[X \to [0,2]^{\leq}]^{is} \overset{lin}{\to} [0,2]^{\leq}]^{pw} \mid \gamma(1) = 1\}$ is isomorphic to $L_2 := \{\beta \in [[X \to [0,2]^{\geq}]^{is} \overset{lin}{\to} [0,2]^{\geq}]^{pw} \mid \beta(1) = 1\}$. But this is more less obvious from the fact that $[0,2]^{\leq}$ and $[0,2]^{\geq}$ are isomorphic as continuous lattices. The isomorphism $T : L_1 \to L_2$ can be given as $T(\gamma)(f) = 2 - \gamma(2 - f)$. Clearly $T \circ T(\gamma) = \gamma$, hence $T = T^{-1}$. We verify that $T(\gamma) \in L_2$ for all $\gamma \in L_1$.

First we observe that if $f \in [X \to [0,2]^{\leq}]$ and f is bounded by $M < 2$, then

$$\begin{aligned}
T(\gamma)(f) &= 2 - \gamma(2 - f) \\
&= (2 - M) + M - \gamma((2 - M) + M - f) \\
&= (2 - M) - \gamma(2 - M) + M - \gamma(M - f) \\
&= (2 - M) - (2 - M)\gamma(1) + M - \gamma(M - f) \\
&= M - \gamma(M - f)
\end{aligned}$$

$T(\gamma)(0) = 2 - 2\gamma(1) = 0$. If f and $rf \in [X \to [0,2]^{\leq}]$ for $r > 0$, then f is bounded by 2 and by $\frac{2}{r}$. By the previous observation it follows that,

$$\begin{aligned}
r \cdot T(\gamma)(f) &= r(\tfrac{2}{r} - \gamma(\tfrac{2}{r} - f)) \\
&= 2 - \gamma(2 - rf) \qquad \text{since } \gamma \text{ is linear in } [0,2] \\
&= T(\gamma)(rf).
\end{aligned}$$

In the same way if $f,\ g,\ f+g \in [X \to [0,2]^{\leq}]$, then $f/2,\ g/2,\ (f+g)/2$ are bounded by 1. Since γ is linear in $[0,2]$,

$$\begin{aligned} T(\gamma)((f+g)/2) &= 2-\gamma(2-(f+g)/2) \\ &= 1-\gamma(1-f/2)+1-\gamma(1-g/2) \\ &= T(\gamma)(f/2)+T(\gamma)(g/2). \end{aligned}$$

From the previous part we conclude that $T(\gamma)(f+g) = 2T(\gamma)((f+g)/2) = T(\gamma)(f)+T(\gamma)(g)$. Finally observe that $T(\gamma)(1) = 2-\gamma(1) = 1$. ■

If we restrict ourselves to valuations $\nu \in \mathbf{V}_+X$, then we can also extend the integrals V^{+l} and V^{+u} to bounded functions $f \in \lceil X \to \mathbb{R}\rceil^{\leq}$ and $g \in \lceil X \to \mathbb{R}\rceil^{\geq}$ as follows. Since $-M_f \leq f$ and $-M_g \leq g$, for all $\nu \in \mathbf{V}_+X$ we set:

- $V^l(f,\nu) := V^{+l}(f+M_f,\nu) - M_f \cdot \nu(X)$.
- $V^u(g,\nu) := V^{+u}(g+M_g,\nu) - M_g \cdot \nu(X)$.

Again it is possible to show that the value of $V^l(f,\nu)$ and $V^u(g,\nu)$ does not depend on which bounding constants M_f and M_g we chose. Since $0 \leq g+M_g \leq 2M_g$, it is easy to check that $V^u(g,\nu) = M_g \cdot \nu(X) - V^l(M_g-g,\nu)$ still holds. Recall from classical analysis that for all $f : X \to \mathbb{R}$ we have that $f = f^+ - f^-$ where $f^+(x) := \max(f(x),0)$ and $f^-(x) = -\min(f(x),0)$ for all $x \in X$. Clearly $f^+, f^- : X \to \mathbb{R}^+$. If f is LSC, then f^+ is LSC but f^- is USC. Conversely if g is USC, then g^+ is USC but g^- is LSC. It is easy to see that the above definitions are equivalent to:

- $V^l(f,\nu) = V^{+l}(f^+,\nu) - V^{+u}(f^-,\nu)$.
- $V^u(g,\nu) = V^{+u}(g^+,\nu) - V^{+l}(g^-,\nu)$.

This shows that if $f \in \lceil X \to \mathbb{R}^{\leq}\rceil$, then $-V^l(f,\nu) = V^u(-f,\nu)$ and similarly if $g \in \lceil X \to \mathbb{R}^{\geq}\rceil$, then $-V^u(g,\nu) = V^l(-g,\nu)$. The following theorems can be verified from the lemmas in this section, Proposition 6.20 and Corollary 6.24. The functions inf and sup in $\mathbb{R}$ are taken with respect to the usual order $\leq$.

Theorem 6.37 *Suppose that $(X,\mathcal{G})$ is a topological space. Then*

1. *$V^l : [X \to [-M,M]^{\leq}]^{is} \times \mathbf{P}^1X \to \mathbb{R}^{\leq}$ is continuous in each argument for every $M \in \mathbb{R}$.*
2. *Fix $\nu \in \mathbf{P}^1X$. Suppose that $\langle f_j\rangle_{j\in J}$ is a monotone net in $\lceil X \to \mathbb{R}^{\leq}\rceil$ and $\sup_j f_j \in \lceil X \to \mathbb{R}^{\leq}\rceil$. Then $V^l(\sup_j f_j,\nu) = \sup_j V^l(f_j,\nu)$.*
3. *Fix $f \in \lceil X \to \mathbb{R}^{\leq}\rceil$. Then $V^l(f,\_) : \mathbf{P}^1X \to \mathbb{R}^{\leq}$ is Scott continuous.*

4. $V^l : \lceil X \to \mathbb{R}^{\leq} \rceil \times \mathbf{P}^1 X \to \mathbb{R}^{\leq}$ *is linear in the first argument and affine*[5] *in the second argument.*

5. *Suppose that* $(Y, \mathcal{H})$ *is a topological space,* $\nu \in \mathbf{P}^1 X$, $h \in [X \to Y]$ *and* $f \in \lceil Y \to \mathbb{R}^{\leq} \rceil$. *Then* $V^l_X(f \circ h, \nu) = V^l_Y(f, \nu_h)$ *where* $\nu_h \equiv \nu \circ h^{-1}$.

6. *The topology* $\mathcal{G}_{pw}(\mathbf{P}^1 X)$ *in* $\mathbf{P}^1 X$ *is the weakest topology such that the function* $V^l(f, \_) : \mathbf{P}^1 X \to \mathbb{R}^{\leq}$ *is Scott continuous (LSC) for all* $f \in \lceil X \to \mathbb{R}^{\leq} \rceil$.

Theorem 6.38 *Suppose that* $(X, \mathcal{G})$ *is a topological space. Then*

1. $V^u : [X \to [-M, M]^{\geq}]^{is} \times \mathbf{P}^1 X \to \mathbb{R}^{\geq}$ *is continuous in each argument for every* $M \in \mathbb{R}$.

2. *Fix* $\nu \in \mathbf{P}^1 X$. *Suppose that* $\langle g_j \rangle_{j \in J}$ *is a monotone net in* $\lceil X \to \mathbb{R}^{\geq} \rceil$ *and* $\inf_j g_j \in \lceil X \to \mathbb{R}^{\geq} \rceil$. *Then* $V^u(\inf_j g_j, \nu) = \inf_j V^u(g_j, \nu)$.

3. *Fix* $g \in \lceil X \to \mathbb{R}^{\geq} \rceil$. *Then* $V^l(g, \_) : \mathbf{P}^1 X \to \mathbb{R}^{\geq}$ *is Scott continuous.*

4. $V^u : \lceil X \to \mathbb{R}^{\geq} \rceil \times \mathbf{P}^1 X \to \mathbb{R}^{\geq}$ *is linear in the first argument and affine in the second argument.*

5. *Suppose that* $(Y, \mathcal{H})$ *is a topological space,* $\nu \in \mathbf{P}^1 X$, $h \in [X \to Y]$ *and* $g \in \lceil Y \to \mathbb{R}^{\geq} \rceil$. *Then* $V^l_X(g \circ h, \nu) = V^l_Y(g, \nu_h)$ *where* $\nu_h \equiv \nu \circ h^{-1}$.

6. *The topology* $\mathcal{G}_{pw}(\mathbf{P}^1 X)$ *in* $\mathbf{P}^1 X$ *is the weakest topology such that the function* $V^l(g, \_) : \mathbf{P}^1 X \to \mathbb{R}^{\geq}$ *is Scott continuous (USC) for all* $g \in \lceil X \to \mathbb{R}^{\geq} \rceil$.

Suppose that $f : X \to \mathbb{R}$ is any bounded function and $\nu \in \mathbf{V}_+ X$. Then $f_E \in [X \to [-M_f, M_f]^{\leq}]$ and $f^E \in [X \to [-M_f, M_f]^{\geq}]$. We say that f is **V-integrable** if

$$V^l(f_E, \nu) = V^u(f^E, \nu)$$

and we denote the common value by $V(f, \nu)$. The following theorem shows that the valuation integral recovers the Lebesgue integral for V-integrable functions.

Theorem 6.39 *Let* $f : X \to \mathbb{R}$ *be any bounded function. If* ν *extends to a probability measure* $\overline{\nu}$, *then*

1. *If* f *is LSC, then* $V^l(f, \nu) = \int f d\overline{\nu}$.

2. *If* f *is USC, then* $V^u(f, \nu) = \int f d\overline{\nu}$.

[5] Linear under convex combinations.

3. *If f is Borel measurable, then $V^l(f_E,\nu) \le \int f d\overline{\nu} \le V^u(f^E,\nu)$.*

4. *f is V-integrable if and only if $\overline{\nu}(D_f) = 0$. In which case $V(f,\nu) = \int f d\overline{\nu}$.*

Proof

1. If $f \in [X \to [-M_f, M_f]^{\le}]$, then $f_E = f$ and $V^l(f,\nu) = V^{+l}(f + M_f,\nu) - M_f \cdot \nu(X)$. By Proposition 6.22 we have that $V^{+l}(f+M_f,\nu) = \int (f+M_f)d\overline{\nu} = \int f d\overline{\nu} + M_f \cdot \nu(X)$ and the claim follows.

2. By the previous part we have that

$$\begin{aligned} V^u(f,\nu) &\equiv -V^l(-f,\nu) \\ &= -\int -f d\overline{\nu} \\ &= \int f d\overline{\nu}. \end{aligned}$$

3. We just have to notice that $f_E \le f \le f^E$ implies that

$$V^l(f_E,\nu) = \int f_E d\overline{\nu} \le \int f d\overline{\nu} \le \int f^E d\overline{\nu} = V^u(f^E,\nu).$$

4. Since $f^E - f_E \ge 0$, it follows that $V^l(f_E,\nu) = V^u(f^E,\nu)$ if and only if $\int (f^E - f_E) d\overline{\nu} = 0$. That is, $\overline{\nu}(D_f) = 0$ by the remarks after Corollary 6.2. The conclusion is immediate from part (4). ■

6.6 The unification theorem

Having set the necessary material we can put all the pieces together to show a unification result which connects the valuation and Riemann-Edalat integrals. The setting is as follows: $(X, \mathcal{G})$ is a Hausdorff topological space and $(D, \sqsubseteq, e, \perp)$ is a pointed S-domain representation for X.

- $\mu \in \mathfrak{M}^1(X;\tau)$. That is, μ is a τ-smooth probability measure on $(X, \mathcal{B}X)$.

- $\mu_\restriction \in \mathbf{P}^1 X$ is the restriction of μ to the paving of open sets $\mathcal{G}$.

- $\mu_e \in \mathbf{P}^1 D$ is the continuous valuation given by $\mu_e(G) = \mu(e^{-1}G) = \mu_\restriction(e^{-1}G)$ for all $G \in \mathcal{G}_\sigma D$. Note that $\mu_e = \bigsqcup^{\uparrow} S(\mu_e)$ where $S(\mu_e) = \{\nu \in \mathbf{P}^1 D \mid \nu \ll \mu_e \text{ and } \nu \text{ simple}\}$.

- $\overline{\mu}_e \in \mathfrak{M}^1(D;\tau)$ is the unique extension of μ_e to a probability measure on $\mathcal{B}D^\sigma$ given by Corollary 2.22. By part (3) of Corollary 6.9 $\overline{\mu}_e$ agrees with the probability measure $E \mapsto \mu(e^{-1}E)$ induced by μ along e (i.e., the law of e).

For a bounded function $f : X \to \mathbb{R}_+$ we have that:

- D_f is the set of discontinuities of the function $f : X \to [-M_f, M_f]^\lambda$.
- $f_E \in [X \to [-M_f, M_f]^{\leq}]$ and $f^E \in [X \to [-M_f, M_f]^{\geq}]$ are the lower and upper semi-continuous envelopes of f defined in Section 6.2.1.
- $f_D \in [D \to [-M_f, M_f]^{\leq}]$ and $f^D \in [D \to [-M_f, M_f]^{\geq}]$ are the greatest continuous extensions of f_E and f^E to D along e constructed in Corollary 6.7 and given by:

  $$\begin{aligned} f_D(d) &:= \sup\{\inf f_E[e^{-1}\Uparrow z] \mid c \ll d \text{ and } c \in D\} \\ f^D(d) &:= \inf\{\sup f^E[e^{-1}\Uparrow z] \mid c \ll d \text{ and } c \in D\} \end{aligned}$$

  Notice then that $f_D = (f_E)_{[*]}$ and $f^D = (f^E)^{[*]}$, the functions defined in Section 6.4.

Theorem 6.40 ***Unification Theorem***

Let $(X, \mathcal{G})$ be a Hausdorff topological space. Let $(D, \sqsubseteq)$ be a pointed continuous dcpo and $e : X \to D^\sigma$ a subspace embedding. Let μ be a τ-smooth probability measure on $\mathcal{B}X$. Then for any bounded function $f : X \to \mathbb{R}$ we have that

$$\begin{aligned} R^u_X(f, \mu) = \int_X f^E d\mu = V^u_X(f^E, \mu_\uparrow) = V^u_D(f^D, \mu_e) = \int_D f^D d\overline{\mu}_e \\ R^l_X(f, \mu) = \int_X f_E d\mu = V^l_X(f_E, \mu_\uparrow) = V^l_D(f_D, \mu_e) = \int_D f_D d\overline{\mu}_e \end{aligned}$$

$\mu(D_f) = 0$ if and only if all the above integrals have the same value, namely $\int_X f d\mu$.

Proof We show the result for $f : X \to \mathbb{R}_+$ bounded non-negative. Notice that

$$\begin{aligned} \int_X f_E d\mu &= V^l_X(f_E, \mu_\uparrow) && \text{by Proposition 6.22} \\ &= V^l_X(f_D \circ e, \mu_\uparrow) && \text{since } f_D \text{ extends } f_E \\ &= V^l_D(f_D, \mu_\uparrow \circ e^{-1}) && \text{by Proposition 6.20-(6)} \\ &= V^l_D(f_D, \mu_e) \\ &= \int_D f_D d\overline{\mu}_e && \text{by Proposition 6.22} \end{aligned}$$

The previous also holds via the usual composition theorem in measure theory (see Section 3.15 in [HJ94] for example).

$$\begin{aligned} V^l_D(f_D, \mu_e) &= \sup_{\nu \in S(\mu_e)} V^l_D(f_D, \nu) && \text{by Scott continuity} \\ &= \sup_{\nu \in S(\mu_e)} \int_D f_D d\overline{\nu} && \text{by Remark 6.21} \\ &= R^l_X(f_E, \mu). && \text{by definition since } (f_E)_{[*]} = f_D \end{aligned}$$

Now suppose that $f : X \to \mathbb{R}$ is any bounded function. Then $(f + M_f)$ is non-negative and we have just shown that,

$$R^l_X(f + M_f, \mu) = \int_X (f + M_f)_E d\mu = V^l_X((f + M_f)_E, \mu_\uparrow) =$$
$$V^l_D((f + M_f)_D, \mu_e) = \int_D (f + M_f)_D d\overline{\mu}_e.$$

But it is trivial to verify that $(f + M_f)_E = f_E + M_f$ and that $(f + M_f)_D = f_D + M_f$. By linearity of the integrals and since $\mu(X) = 1$ we have that

- $R^l_X(f + M_f, \mu) = R^l_X(f, \mu) + M_f$
- $\int_X (f + M_f)_E d\mu = \int_X f_E + M_f$.
- $V^l_X((f + M_f)_E, \mu_\uparrow) = V^l_X(f_E, \mu_\uparrow) + M_f$
- $V^l_D((f + M_f)_D, \mu_e) = V^l_D(f_D, \mu_e) + M_f$
- $\int_D (f + M_f)_D d\overline{\mu}_e = \int_D f_D d\overline{\mu}_e + M_f$

and the claim follows. For the case of the upper integral the proof is similar. ■

Remark 6.41 *The reader should have in mind that if f is LSC, then $f = f_E$; if f is USC, then $f = f^E$; and if f is continuous (in the metric topology of $[-M_f, M_f]$), then $f = f_E = f^E$.*

7. Probabilistic domains

7.1 Summary

The topics of this chapter are the probabilistic powerdomains $\mathbf{P}^1X$ and $\mathbf{P}X$ of a topological space $(X, \mathcal{G})$. In this chapter we make the following convention. Continuous valuations in $\mathbf{V}X$ are denoted by Greek letters: μ, ν, etc.; τ-smooth measures in $\mathfrak{M}(X;\tau)$ are denoted with overlined Greek letters: $\overline{\mu}$, $\overline{\nu}$. It is understood that $\mu = \overline{\mu} \restriction_{\mathcal{G}}$. Suppose that $(Y, \mathcal{H})$ is a topological space, $e \in [X \to Y]$ and $\mu \in \mathbf{V}_+X$. By an argument similar to the one used in part (3) of Corollary 6.9 we have that:

- The function μ_e given by $\mu_e(H) = \mu(e^{-1}H)$ for all $H \in \mathcal{H}$ is a finite continuous valuation on Y. That is, $\mu_e \in \mathbf{V}_+Y$.
- The function $\overline{\mu}_e$ given by $\mu_e(E) = \overline{\mu}(e^{-1}E)$ for all $E \in \mathcal{B}Y$ is a finite τ-smooth Borel measure on Y. That is, $\overline{\mu}_e \in \mathfrak{M}_+(Y;\tau)$.
- Since $\mathcal{H}$ is a π-system, then by part (3) of Proposition 1.23, $\overline{\mu}_e$ is the only possible extension of μ_e to a measure $\overline{\mu_e} \in \mathfrak{M}_+(Y;\tau)$.

We use the mathematical machinery developed in Chapter 6 to show three important results. Namely,

- (Theorem 7.10) Suppose that $(X, \mathcal{G})$ is any topological space, $(D, \sqsubseteq)$ is a continuous dcpo and the map $e : X \to D$ embeds X as subspace of D where D is endowed with the Scott topology. In other words, $(D, \sqsubseteq, e)$ is an S-domain representation for X (Definition 5.1). Then a net $(\overline{\mu}^j)_{j\in J}$ in $\mathfrak{M}^1(X;\tau)$ converges in the weak topology to $\overline{\mu} \in \mathfrak{M}^1(X;\tau)$, if and only if, the net $(\mu_e^j)_{j\in J}$ converges to μ_e in the Scott topology of $\mathbf{P}^1D$.
- (Corollary 7.17) Suppose that $(X, \mathcal{G})$ is a stably locally compact space. Let ν and μ be normalised continuous valuations on X, and let $\overline{\nu}$ and $\overline{\mu}$ denote their unique extensions to Radon probability measures on $(X, \sigma(\mathcal{G}_p))$ respectively. Then $\nu \sqsubseteq \mu$ if and only if $\overline{\nu}$ is stochastically smaller than $\overline{\mu}$. In which case there exists a continuous valuation $\lambda \in \mathbf{P}^1(X \times X)$ with $\pi_1[\lambda] = \nu$, $\pi_2[\lambda] = \mu$ and $\overline{\lambda}(\{(x, y) \mid x \sqsubseteq_s y\}) = 1$.

- (Theorems 7.26 and 7.27) Suppose that $(X, \mathcal{G})$ is a stably compact space. Then $\mathbf{P}^1 X$ and $\mathbf{P}X$ can be given topologies coarser than the Scott topology which make them stably compact spaces. This solves a question raised by Jung and Tix in [JT98].

The agreement of the Scott and pointwise convergence topologies with the weak topology was studied by Kirch [Kir93], Tix [Tix95] and Heckmann [Hec96] in the context of the valuation integral. Edalat [Eda97b] and later Lawson [Law98] obtained other characterisations relating the weak and Scott topologies for the R-integral. The characterisation presented here is one of the important consequences of the unification theorem (Theorem 6.40). The second result above is a generalisation of the splitting lemma (Lemma 2.4) stated in Chapter 2. The splitting lemma was rediscovered by Jones [Jon89] in the context of probabilistic non-determinism, but it is an instance of the more general problem of finding probability distributions with given marginals. The latter problem has been the object of intense study in probability theory (see Section 7.5 below). The third result above solves an open question raised by Jung and Tix in [JT98] who showed a similar result for stably locally compact domains.

At the end of the chapter we review part of the theory of metrics in spaces of measures and suggest some directions for future work.

7.2 A review of topologies in spaces of measures

Our exposition is extracted from Topsøe [Top70b] and Schwartz [Sch73]. All the results are stated in terms of net convergence. As we know from Remark 1.8 there are certain pitfalls to be avoided when working with subnets. More recently Dal Maso, *et al* [MGM87] have shown that for Hausdorff spaces the theory can be reconstructed using only convergence of sequences. Recall that for a topological space $(X, \mathcal{G})$ the set of finite Borel measures on $(X, \mathcal{B}X)$ is denoted as $\mathfrak{M}_+(X)$. If $\overline{\mu} \in \mathfrak{M}_+(X)$, a Borel measurable function $f : X \to \mathbb{R}$ is called a **$\overline{\mu}$-continuity function** if $\overline{\mu}(D_f) = 0$. A Borel set $E \in \mathcal{B}X$ is called a **$\overline{\mu}$-continuity set** if $\overline{\mu}(\partial E) = 0$ where $\partial E \equiv cl(E) \cap cl(E^{\mathrm{c}})$ is the boundary of E.

Definition 7.1 *Let $(X, \mathcal{G})$ be a topological space. The* ***weak topology*** *(also known as the topology of* ***narrow convergence****) $\mathcal{G}_w(\mathfrak{M}X)$ is the weakest topology on $\mathfrak{M}_+(X)$ such that for every $f : X \to \mathbb{R}$ bounded lower semi-continuous, the function $\overline{\mu} \mapsto \int f d\overline{\mu}$ is lower semi-continuous. If a net $(\overline{\mu}_j)_{j\in J}$ in $\mathfrak{M}_+(X)$ converges to $\overline{\mu}$ in the weak topology, we write $\overline{\mu}_j \to_w \overline{\mu}$.*

We note that the usual definition in topological measure theory (as in [Top70b]) requires $(X, \mathcal{G})$ to be Hausdorff. We have the following proposition.

Proposition 7.2 ***Topsøe's portmanteau[1] theorem.***

Suppose that $(X, \mathcal{G})$ is a Hausdorff space and $(\overline{\mu}_j)_{j \in J}$ is a net in $\mathfrak{M}_+(X)$. The following five conditions are equivalent:

(*i*) $\overline{\mu}_j \to_w \overline{\mu}$.

(*ii*) $\int f d\overline{\mu} \leq \liminf_j \int f d\overline{\mu}_j$ *for all* $f \in \lceil X \to \mathbb{R}^{\leq} \rceil$. *That is, f bounded LSC.*

(*iii*) $\limsup_j \int g d\overline{\mu}_j \leq \int g d\overline{\mu}$ *for all* $g \in \lceil X \to \mathbb{R}^{\geq} \rceil$. *That is, g bounded USC.*

(*iv*) $\overline{\mu}(G) \leq \liminf_j \overline{\mu}_j(G)$ *for all* $G \in \mathcal{G}$ *and* $\overline{\mu}(X) = \lim_j \overline{\mu}_j(X)$.

(*v*) $\limsup_j \overline{\mu}_j(F) \leq \overline{\mu}(F)$ *for all* $F \in \mathcal{F}$ *and* $\overline{\mu}(X) = \lim_j \overline{\mu}_j(X)$.

Each of them implies the following three conditions.

(*vi*) $\overline{\mu}(E) = \lim_j \overline{\mu}_j(E)$ *for all $\overline{\mu}$-continuity sets E.*

(*vii*) $\int f d\overline{\mu} = \lim_j \int f d\overline{\mu}_j$ *for all bounded $\overline{\mu}$-continuity functions* $f : X \to \mathbb{R}$.

(*viii*) $\int f d\overline{\mu} = \lim_j \int f d\overline{\mu}_j$ *for all* $f \in \lceil X \to \mathbb{R}^{\lambda} \rceil$. *That is, for all f bounded and continuous with respect to the metric topology of* $\mathbb{R}$.

If $(X, \mathcal{G})$ is $T_{3\frac{1}{2}}$ (i.e., completely regular) and $\overline{\mu}$ is τ-smooth, then the eight conditions are equivalent. Furthermore if $\mathcal{U}$ is a uniformity on X, then we can add the following equivalence to the list:

(*ix*) $\int f d\overline{\mu} = \lim_j \int f d\overline{\mu}_j$ *for all* $f : X \to \mathbb{R}$ *bounded and uniformly continuous with respect to $\mathcal{U}$.*

Proof See Theorem 8.1 in [Top70b]. ■

From condition (*iv*) it is immediate that if $(\overline{\mu}_j)_{j \in J}$ is a net in $\mathfrak{M}_+(X; \tau)$ and $\overline{\mu} \in \mathfrak{M}_+(X; \tau)$ then,

$$\overline{\mu}_j \to_w \overline{\mu} \text{ implies that } \mu_j \to_{pw} \mu \text{ in } \mathbf{V}_+ X.$$

[1] Because of the importance of this result it is said that it should always be carried, which explains its name: in English the word "portmanteau" is used as a synonym for large traveling suitcase (not the name of a mathematician as some authors believe!) Curiously enough, in French, portmanteau means coat stand or coat hanger.

Which shows that the map $(\_)_{\restriction} : \mathfrak{M}_+(X;\tau) \to \mathbf{V}_+X$ given by $\overline{\mu}_{\restriction} := \overline{\mu} \restriction_{\mathcal{G}} = \mu$ is continuous (Proposition 1.9-(4)). By Proposition 1.23-(3) the map $(\_)_{\restriction}$ is injective. Obviously $(\_)_{\restriction}$ will be onto if every finite continuous valuation on X extends (uniquely) to a measure; but even in this case, if we wanted $(\_)_{\restriction}$ to be a homeomorphism, the condition $\overline{\mu}(X) = \lim_j \overline{\mu}_j(X)$ forces us to restrict ourselves to $\mathfrak{M}^r(X;\tau)$ for some $r \in (0,\infty)$. Informally, this means the topology of weak convergence in $\mathfrak{M}_+(X;\tau)$ has more open sets than $\mathcal{G}_{pw}(\mathbf{V}_+X)$. This situation is related to the problem that we had in Section 6.5 to ensure that the upper integral $V^{+u}(\_,\_)$ was monotone in the second argument; recall the statement of Lemma 6.32 and the remarks following Lemma 6.34. In Section 7.6 we show that if $(X,\mathcal{H},\leq)$ is a compact pospace space, then $\mathfrak{M}^1(X;\tau)$ and $\mathfrak{M}^{\leq 1}(X;\tau)$ with the weak topology and a suitable order are also compact pospaces. This will enable us to conclude that if $(X,\mathcal{G})$ is a stably compact space, then $(\mathbf{P}^1X, \mathcal{G}_{pw}(\mathbf{P}^1X))$ and $(\mathbf{P}X, \mathcal{G}_{pw}(\mathbf{P}X))$ are stably compact spaces.

The weak topology has the following properties. Recall that $\mathfrak{M}_+(X;R)$ and $\mathfrak{M}_+(X;\tau)$ are the sets of finite Radon and finite τ-smooth measures on $(X,\mathcal{B}X)$.

Proposition 7.3 *Suppose that $(X,\mathcal{G})$ is a Hausdorff space. Then,*

1. *$\mathfrak{M}_+(X;R)$ is a Hausdorff space.*
2. *If X is regular (T_3), then $\mathfrak{M}_+(X;\tau)$ is a Hausdorff space.*
3. *The function $\boldsymbol{\delta} : X \to \mathfrak{M}^1(X)$ given by $\boldsymbol{\delta}(x) := \delta_x$ is a subspace embedding.*
4. *$\boldsymbol{\delta}[X]$ is a closed subset of $\mathfrak{M}^1(X;\tau)$.*
5. *If X is separable, then $\mathfrak{M}_+(X)$ is so.*
6. *X is second countable if and only if $\mathfrak{M}_+(X;\tau)$ is so.*
7. *X is regular if and only if $\mathfrak{M}_+(X;\tau)$ is so.*
8. *X is completely regular ($T_{3\frac{1}{2}}$) if and only if $\mathfrak{M}_+(X;\tau)$ is so.*
9. *X is compact if and only if $\mathfrak{M}^1(X;\tau)$ is so.*

Proof See Appendix in [Sch73]; Theorems 11.1, 11.2 and remarks in p. 76 of [Top70b]. ■

Now observe that the weak topology could also have been defined as the weakest topology in $\mathfrak{M}_+(X)$ such that a net $(\overline{\mu}_j)_{j\in J}$ in $\mathfrak{M}_+(X)$ converges to $\overline{\mu}$, if and only if,

- $\overline{\mu}(G) \leq \liminf_j \overline{\mu}_j(G)$ for all $G \in \mathcal{G}$.
- $\limsup_j \overline{\mu}_j(F) \leq \overline{\mu}(F)$ for all $F \in \mathcal{F}$.

The reason is that X is closed and open, hence $\overline{\mu}(X) = \lim_j \overline{\mu}_j(X)$. This remark should help to put in perspective the following definition:

Definition 7.4 *[Top70b, Section 7] Let $(X, \mathcal{G})$ be a Hausdorff space. The* ***vague topology*** *$\mathcal{G}_{va}(\mathfrak{M}X)$ is the weakest topology on $\mathfrak{M}^{\leq 1}(X; R)$ such that a net $(\overline{\mu}_j)_{j \in J}$ in $\mathfrak{M}^{\leq 1}(X; R)$ converges to $\overline{\mu} \in \mathfrak{M}^{\leq 1}(X; R)$, if and only if,*

- $\overline{\mu}(G) \leq \liminf_j \overline{\mu}_j(G)$ *for all* $G \in \mathcal{G}$*, and*
- $\limsup_j \overline{\mu}_j(K) \leq \overline{\mu}(K)$ *for all* $K \in \mathcal{K}$*.*

For locally compact spaces this definition is equivalent to $\int f d\overline{\mu} = \lim_j \int f d\overline{\mu}_j$ for all $f \in [X \to \mathbb{R}^\lambda]$ such that f has compact support. It is clear that $\mathcal{G}_{va}(\mathfrak{M}X) \subseteq \mathcal{G}_w(\mathfrak{M}X)$ and if X is compact Hausdorff, then they agree. We have the following properties.

Proposition 7.5 *Suppose that $(X, \mathcal{G})$ is a Hausdorff space. Then,*

1. *The function $\boldsymbol{\delta} : X \to \mathfrak{M}^{\leq 1}(X; R)$ given by $\boldsymbol{\delta}(x) := \delta_x$ is a subspace embedding.*
2. *$(\mathfrak{M}^{\leq 1}(X; R), \mathcal{G}_{va}(\mathfrak{M}X))$ is a compact space (not necessarily Hausdorff).*
3. *If X is locally compact, then $(\mathfrak{M}^{\leq 1}(X; R), \mathcal{G}_{va}(\mathfrak{M}X))$ is Hausdorff.*

Proof See Theorem 7.7 in [Top70b]. ■

It follows that if X is locally compact Hausdorff, then $(\mathfrak{M}^{\leq 1}(X; R), \mathcal{G}_{va}(\mathfrak{M}X))$ is a compact Hausdorff space. By Proposition 1.28-(3) we know that if X is locally compact Hausdorff, then $\mathfrak{M}^{\leq 1}(X; R) = \mathfrak{M}^{\leq 1}(X; \tau)$. So one might ask if for such a space $(\mathfrak{M}^{\leq 1}(X; \tau), \mathcal{G}_w(\mathfrak{M}X))$ is compact Hausdorff. The answer to this question is settled by the following corollary.

Corollary 7.6 *Suppose that $(X, \mathcal{G})$ is a locally compact Hausdorff space. Then $(\mathfrak{M}^{\leq 1}(X; \tau), \mathcal{G}_w(\mathfrak{M}X))$ is compact Hausdorff if and only if X is compact.*

Proof

$\boxed{\Leftarrow}$ We noted that for compact Hausdorff spaces $\mathfrak{M}^{\leq 1}(X; \tau) = \mathfrak{M}^{\leq 1}(X; R)$. By parts (2) and (3) of Proposition 7.5 we have that $(\mathfrak{M}^{\leq 1}(X; R), \mathcal{G}_{va}(\mathfrak{M}X))$

is a compact Hausdorff space. But if X is compact Hausdorff, then $\mathcal{G}_{va}(\mathfrak{M}X) = \mathcal{G}_w(\mathfrak{M}X)$.

$\boxed{\Rightarrow}$ We claim that $\mathfrak{M}^1(X;\tau)$ is closed in the weak topology of $\mathfrak{M}^{\leq 1}(X;\tau)$. Suppose that $(\overline{\mu}_j)_{j\in J}$ is a net in $\mathfrak{M}^1(X;\tau)$ and $\overline{\mu} \in \mathfrak{M}^{\leq 1}(X)$. If $\overline{\mu}_j \to_w \overline{\mu}$, then by condition (iv) in Proposition 7.13 we have that $\overline{\mu}(X) = \lim_j \overline{\mu}_j(X) = 1$. Hence $\overline{\mu} \in \mathfrak{M}^1(X;\tau)$. By Proposition 1.9-(1) it follows that $\mathfrak{M}^1(X;\tau)$ is closed.

On the other hand, by parts (3) and (4) of Proposition 7.3, we have that $(X, \mathcal{G})$ is homeomorphic to $\boldsymbol{\delta}[X]$ and $\boldsymbol{\delta}[X]$ is closed in the weak topology of $\mathfrak{M}^1(X;\tau)$. Hence $X \cong \boldsymbol{\delta}[X]$ is a closed subset of $\mathfrak{M}^{\leq 1}(X;\tau)$. Part (2) of Proposition 7.3 implies that $(\mathfrak{M}^{\leq 1}(X;\tau), \mathcal{G}_w(\mathfrak{M}X))$ is Hausdorff since locally compact Hausdorff spaces are regular. We conclude that $(X, \mathcal{G})$ is compact. ■

The previous result shows that in general the vague topology has less opens than the weak topology. We will not use the vague topology any further; so for the rest of the chapter we assume that $\mathfrak{M}_+(X)$ is endowed with the weak topology.

7.3 Agreement of the Scott topology and the weak topology

Suppose that $(D, \sqsubseteq)$ is a continuous dcpo. Kirch [Kir93, Satz 8.6] and Tix [Tix95, Satz 4.10] showed that the Scott topology on $\mathbf{V}D$ is the weakest topology on $\mathbf{V}D$ such that the map $\nu \mapsto V^{+l}(f, \nu)$ is Scott continuous for all $f \in [D \to [0,\infty]^{\leq}]$. Heckmann [Hec96, Theorem 8.3] gave a similar result for the pointwise convergence topology $\mathcal{G}_{pw}(\mathbf{V}X)$ in $\mathbf{V}X$ for any topological space X (recall from Remark 6.17 that $\mathcal{G}_{pw}(\mathbf{V}D)$ agrees with the Scott topology on $\mathbf{V}D$).

Suppose that (X, d) is a separable metric space and there exists an S-domain representation $(D, \sqsubseteq, e)$ for X, such that $e[X]$ is a G_δ subset of D. Suppose that $\mathfrak{M}^1(X)$ is endowed with the weak topology and $\mathbf{P}^1 D$ is endowed (as usually) with the Scott topology. Edalat [Eda97b] showed that the function $\overline{\mu} \mapsto \mu_e$ from $\mathfrak{M}^1(X)$ to $\mathbf{P}^1 D$ is a subspace embedding. Hence, if $(\overline{\mu}_j)_{j\in J}$ is a net in $\mathfrak{M}^1(X)$ and $\overline{\mu} \in \mathfrak{M}^1(X)$, then

$$\overline{\mu}_j \to_w \overline{\mu} \text{ if and only if } (\mu_j)_e \to_\sigma \mu_e.$$

Lawson [Law98, Theorem 8.7] obtained a similar result for Polish spaces and (ω-continuous) MP-hulls. Our aim is to obtain a general equivalence between the weak topology in the space of measures of any topological space, and the Scott topology of the probabilistic powerdomain of any S-domain representation for the space. We begin by proving two technical lemmas. The following lemma shows that the first part of Proposition 7.2 remains valid for the space of probability measures of any topological space.

Lemma 7.7 *Let $(X, \mathcal{G})$ be any topological space. Suppose that $(\overline{\mu}_j)_{j\in J}$ is a net in $\mathfrak{M}^1(X)$ and $\overline{\mu} \in \mathfrak{M}^1(X)$. The following conditions are equivalent:*

1. $\overline{\mu}_j \to_w \overline{\mu}$.
2. $\int f d\overline{\mu} \leq \liminf_j \int f d\overline{\mu}_j$ *for all* $f \in \lceil X \to \mathbb{R}^{\leq} \rceil$.
3. $\limsup_j \int g d\overline{\mu}_j \leq \int g d\overline{\mu}$ *for all* $g \in \lceil X \to \mathbb{R}^{\geq} \rceil$.
4. $\overline{\mu}(G) \leq \liminf_j \overline{\mu}_j(G)$ *for all* $G \in \mathcal{G}$.
5. $\limsup_j \overline{\mu}_j(F) \leq \overline{\mu}(F)$ *for all* $F \in \mathcal{F}$.

Proof (1) $\Leftrightarrow$ (2) follows by definition. (2) $\Leftrightarrow$ (3) is obvious since $f \in \lceil X \to \mathbb{R}^{\leq} \rceil$ if and only if $-f \in \lceil X \to \mathbb{R}^{\geq} \rceil$. (4) $\Leftrightarrow$ (5) is obvious since $\overline{\mu}(F) = 1 - \overline{\mu}(F^c)$ for all $F \in \mathcal{F}$ and $\overline{\mu} \in \mathfrak{M}^1(X)$. (2) $\Rightarrow$ (4) is trivial since for every $G \in \mathcal{G}$ we have that $\mathbb{I}_G \in \lceil X \to \mathbb{R}^{\leq} \rceil$ hence

$$\overline{\mu}(G) = \int \mathbb{I}_G d\overline{\mu} \leq \liminf_j \int \mathbb{I}_G d\overline{\mu}_j = \liminf_j \overline{\mu}_j(G).$$

For proving (4) $\Rightarrow$ (2), note that the inequality above shows that for all simple functions f of the form $f = \sum_{i=1}^n r_i \, \mathbb{I}_{G_i}$ with $r_i \in \mathbb{R}$ we have that $\int f d\overline{\mu} \leq \liminf_j \int f d\overline{\mu}_j$. Now chose any $f \in \lceil X \to \mathbb{R}^{\leq} \rceil$ and let (f_n) be the sequence of simple functions constructed in Lemma 6.3. Then,

$$\begin{aligned} &\int f_n d\overline{\mu}_j \leq \sup_n \int f_n d\overline{\mu}_j = \int f d\overline{\mu}_j \\ &\qquad\qquad \Rightarrow \liminf_j \int f_n d\overline{\mu}_j \leq \liminf_j \int f d\overline{\mu}_j \ \forall n \in \mathbb{N} \\ &\qquad\qquad \Rightarrow \sup_n(\liminf_j \int f_n d\overline{\mu}_j) \leq \liminf_j \int f d\overline{\mu}_j. \end{aligned}$$

Hence

$$\begin{aligned} \int f d\overline{\mu} &= \sup_n \int f_n d\overline{\mu} \\ &\leq \sup_n(\liminf_j \int f_n d\overline{\mu}_j) \\ &\leq \liminf_j \int f d\overline{\mu}_j \end{aligned}$$

and the conclusion follows. ■

Lemma 7.8 *Suppose that $(X, \mathcal{G})$ is any topological space.*

1. *The set $\mathfrak{M}^1(X;\tau)$ topologised with the weak topology $\mathcal{G}_w(\mathfrak{M}X)$ is a subspace of $\mathbf{P}^1X$ topologised with the pointwise convergence topology $\mathcal{G}_{pw}(\mathbf{P}^1X)$.*

2. *If every continuous valuation $\nu \in \mathbf{P}^1X$, extends (uniquely) to a measure $\overline{\nu} \in \mathfrak{M}^1(X;\tau)$, then $(\mathfrak{M}^1(X;\tau), \mathcal{G}_w(\mathfrak{M}X))$ and $(\mathbf{P}^1X, \mathcal{G}_{pw}(\mathbf{P}^1X))$ are homeomorphic.*

3. *Suppose that $(Y,\mathcal{H})$ is any topological space and $e : X \to Y$ is a subspace embedding. Then the function*

$$\mathbf{g} : \mathbf{P}^1X \to \mathbf{P}^1Y$$

given by $\mathbf{g}(\mu) = \mu_e$ is a subspace embedding.

4. *Suppose that $(Y,\mathcal{H})$ is a core-compact space and $e : X \to Y$ is a subspace embedding. Then $\mathbf{e} :(\mathfrak{M}^1(X;\tau), \mathcal{G}_w(\mathfrak{M}X)) \to (\mathbf{P}^1Y)^\sigma$ given by $\mathbf{e}(\overline{\mu}) := \mu_e$ is a subspace embedding.*

Proof

$\boxed{1}$ We claim that the restriction map $(\_)_\restriction : \mathfrak{M}^1(X;\tau) \to \mathbf{P}^1X$ given by $\overline{\mu}_\restriction = \overline{\mu} \restriction_{\mathcal{G}} = \mu$ is a subspace embedding. We noted in Section 7.2 that $(\_)_\restriction$ is injective and continuous when X is Hausdorff. In the light of the equivalence (1) $\Leftrightarrow$ (3) in Lemma 7.7, the same arguments can be repeated to show that $(\_)_\restriction$ is injective and continuous for any topological space. Set $\mathcal{E} := \{\nu \in \mathbf{P}^1X \mid \nu \text{ extends to a Borel measure } \overline{\nu}\}$. Then $\mathcal{E}$ is clearly the image of $\mathfrak{M}^1(X;\tau)$ under $(\_)_\restriction$.

We claim that $(\_)_\restriction^{-1} : \mathcal{E} \to \mathfrak{M}^1(X;\tau)$ is continuous. Suppose that $(\nu_j)_{j\in J}$ is a net in $\mathcal{E}$ converging pointwise to $\nu \in \mathcal{E}$. Then $\overline{\nu}(G) \leq \liminf_j \overline{\nu}_j(G)$ for all $G \in \mathcal{G}$ and by the equivalence (1) $\Leftrightarrow$ (4) in Lemma 7.7 it follows that $(\nu_j)_\restriction^{-1} = \overline{\nu}_j \to_w \overline{\nu} = (\nu)_\restriction^{-1}$. By Proposition 1.9-(4) we have that $(\_)_\restriction^{-1}$ is continuous.

$\boxed{2}$ Notice that the map $(\_)_\restriction : \mathfrak{M}^1(X;\tau) \to \mathbf{P}^1X$ is onto, hence the claim follows from part (1).

$\boxed{3}$ Suppose that $\mu, \nu \in \mathbf{P}^1X$. If $\mu \neq \nu$, then there exists an open set $G \in \mathcal{G}$ such that $\mu(G) \neq \nu(G)$. Since e is a subspace embedding there exists $H \in \mathcal{H}$ with $G = e^{-1}H$. Hence $\mathbf{g}(\mu)(H) = \mu(e^{-1}H) = \mu(G) \neq \nu(G) = \mathbf{g}(\nu)(H)$, which shows that $\mathbf{g}$ is injective. Suppose that $(\mu_j)_{j\in J}$ is a net in $\mathbf{P}^1X$ and $\mu \in \mathbf{P}^1X$. Then since e is a subspace embedding we have that

$$\begin{aligned}\mu_j \to_{pw} \mu \quad &\Leftrightarrow \mu(G) \leq \liminf\nolimits_j \mu_j(G) \text{ for all } G \in \mathcal{G}\\ &\Leftrightarrow \mathbf{g}(\mu)(H) \leq \liminf\nolimits_j \mathbf{g}(\mu_j)(H) \text{ for all } H \in \mathcal{H}\\ &\Leftrightarrow \mathbf{g}(\mu_j) \to_{pw} \mathbf{g}(\mu).\end{aligned}$$

Which shows that $\mathbf{g}$ and $\mathbf{g}^{-1} : \mathbf{g}[\mathbf{P}^1X] \to \mathbf{P}^1X$ are continuous.

$\boxed{4}$ By parts (1) and (3) we know that the functions

$$\begin{aligned}(\_)\restriction \quad &: (\mathfrak{M}^1(X;\tau), \mathcal{G}_w(\mathfrak{M}X)) \to (\mathbf{P}^1X, \mathcal{G}_{pw}(\mathbf{P}^1X))\\ \mathbf{g} \quad &: (\mathbf{P}^1X, \mathcal{G}_{pw}(\mathbf{P}^1X)) \to (\mathbf{P}^1Y, \mathcal{G}_{pw}(\mathbf{P}^1Y))\end{aligned}$$

are subspace embeddings, therefore $\mathbf{e} := \mathbf{g} \circ (\_)\restriction : (\mathfrak{M}^1(X;\tau), \mathcal{G}_w(\mathfrak{M}X)) \to (\mathbf{P}^1Y, \mathcal{G}_{pw}(\mathbf{P}^1Y))$ is a subspace embedding. But since Y is core-compact, by Corollary 6.16 (and remarks following) we have that $\mathcal{G}_{pw}(\mathbf{P}^1Y)$ agrees with the Scott topology in $\mathbf{P}^1Y$. ■

Remark 7.9 *From Proposition 6.15 it follows that part (4) of the previous lemma should hold for any space $(Y, \mathcal{H})$ such that the lattice $(\mathcal{H}, \subseteq)$ (endowed with the Scott topology) is a locally finite-bottomed space. The author does not know any nice characterisation of such spaces.*

Now we are able to show the main result of the section.

Theorem 7.10 ***Scott topology = weak topology (for S-domain representations)***

Suppose that $(X, \mathcal{G})$ is any topological space, $(D, \sqsubseteq)$ is a continuous dcpo and $e : X \to D^\sigma$ is a subspace embedding. Then, the function $\mathbf{e} : \mathfrak{M}^1(X;\tau) \to (\mathbf{P}^1D)^\sigma$ given by $\mathbf{e}(\overline{\mu}) = \mu_e$ is a subspace embedding. In particular for every net $(\overline{\mu}_j)_{j\in J}$ in $\mathfrak{M}^1(X;\tau)$ and $\overline{\mu} \in \mathfrak{M}^1(X;\tau)$ we have that

$$\overline{\mu}_j \to_w \overline{\mu} \textit{ if and only if } \mathbf{e}(\mu_j) \to_\sigma \mathbf{e}(\mu).$$

Proof This is immediate from part (4) of the previous lemma since continuous dcpo's are locally compact and sober spaces (Proposition 1.10), and we know that in sober spaces core-compactness and local compactness are equivalent (Section 6.2.4). ■

7.4 A useful construction

The following construction was defined in the proof of Corollary 3.40 and is key for the results in the following sections. Suppose that $(X, \mathcal{G})$ is a stably *locally* compact space, set $X_\perp := X \cup \{\perp\}$ and $\mathcal{G}_\perp := \mathcal{G} \cup \{X_\perp\}$. Let $cl_\perp(E)$ denote the closure in $\mathcal{G}_\perp$ of a set $E \subseteq X_\perp$. Since for all $x \in X_\perp$ we have that $(cl_\perp(x))^\mathsf{c} \in \mathcal{G}$, it follows that $\perp \notin (cl_\perp(x))^\mathsf{c}$. Hence, $\perp \in cl_\perp(x)$ for all $x \in X_\perp$; that is, $\perp$ is the least element with respect to the specialisation order in $X_\perp$. It is easy to verify that $(X_\perp, \mathcal{G}_\perp)$ is sober and $\mathcal{Q}_\perp = \mathcal{Q} \cup \{X_\perp\}$. Hence

- $(X_\perp, \mathcal{G}_\perp)$ is a stably compact space.

- Any $\nu \in \mathbf{V}_+X$ extends to $\nu^\perp \in \mathbf{V}_+X_\perp$ by setting
$$\nu^\perp(G) = \nu(G) \text{ for all } G \in \mathcal{G} \text{ and } \nu^\perp(X_\perp) = \nu(X).$$
For all $\nu, \mu \in \mathbf{V}_+X$ we have that $\nu \sqsubseteq \mu$ if and only if $\nu^\perp \sqsubseteq \mu^\perp$.

- Similarly for any $\nu \in \mathbf{P}X$, we define $\nu^1 \in \mathbf{P}^1X_\perp$ as
$$\nu^1(G) = \nu(G) \text{ for all } G \in \mathcal{G} \text{ and } \nu^1(X_\perp) = 1.$$
For all $\nu, \mu \in \mathbf{P}X$ we have that $\nu \sqsubseteq \mu$ if and only if $\nu^1 \sqsubseteq \mu^1$.

As a consequence of Proposition 1.13 we obtain an equivalent construction for any compact pospace $(X, \mathcal{H}, \leq)$. Let $\mathcal{G} := \mathcal{U}_\leq \equiv \{G \in \mathcal{H} \mid G = \uparrow G\}$, then $(X, \mathcal{G})$ is a stably compact space and we can apply the previous construction to obtain $(X_\perp, \mathcal{G}_\perp)$. Define $\mathcal{H}_\perp$ as the patch topology of $\mathcal{G}_\perp$. The reader can verify the following:

- $(X_\perp, \mathcal{H}_\perp, \leq)$ is a compact pospace where $x \leq y$ if and only if $x = \perp$ or $x, y \in X$ and $x \leq y$.
- X and $\{\perp\}$ are closed and open (clopen) subsets of $(X_\perp, \mathcal{H}_\perp)$. This is because, X is a compact saturated subset of $(X_\perp, \mathcal{G}_\perp)$, hence by Lemma 3.12-(2) X is a compact subset of the Hausdorff space $(X_\perp, \mathcal{H}_\perp)$.
- $\mathcal{H}_\perp = \mathcal{T}(\mathcal{H} \cup \{\{\perp\}\}) = \mathcal{H} \cup \{G \cup \{\perp\} \mid G \in \mathcal{H}\}$.

If $\nu \in \mathbf{V}_+X$, then $\overline{\nu^\perp}(\{\perp\}) = \overline{\nu^\perp}(X_\perp) - \overline{\nu^\perp}(X) = 0$. Therefore any $\overline{\nu} \in \mathfrak{M}_+(X;\tau)$ extends to $\overline{\nu}^\perp \in \mathfrak{M}_+(X_\perp;\tau)$ by setting $\overline{\nu}^\perp := \overline{\nu^\perp}$. Similarly any $\overline{\mu} \in \mathfrak{M}^{\leq 1}(X;\tau)$ extends to $\overline{\mu}^1 \in \mathfrak{M}^1(X_\perp;\tau)$ by setting $\overline{\mu}^1 := \overline{\mu^1}$ and we have that $\overline{\mu}^1(\{\perp\}) = 1 - \overline{\mu}(X)$.

Lemma 7.11 *Suppose that $(X, \mathcal{H}, \leq)$ is a compact pospace, $(\overline{\mu}_j)_{j\in J}$ is a net in $\mathfrak{M}^{\leq 1}(X;\tau)$ and $\overline{\mu} \in \mathfrak{M}^{\leq 1}(X;\tau)$. Then*
$$\overline{\mu}_j \to_w \overline{\mu} \textit{ if and only if } \overline{\mu}_j^1 \to_w \overline{\mu}^1 \textit{ in } \mathfrak{M}^1(X_\perp;\tau).$$
Hence $\mathfrak{M}^{\leq 1}(X;\tau)$ is a subspace of $\mathfrak{M}^1(X_\perp;\tau)$.

Proof $\boxed{\Rightarrow}$ Suppose that $G \in \mathcal{H}$, then by condition (iv) in Proposition 7.2 we have that: $\overline{\mu}^1(G) \leq \liminf_j \overline{\mu}_j^1(G)$ since $\overline{\nu}^1$extends $\overline{\nu}$ for all $\overline{\nu} \in \mathfrak{M}^{\leq 1}(X;\tau)$. Now suppose that $G = H \cup \{\perp\}$ with $H \in \mathcal{H}$. Since $\overline{\nu}^1(\{\perp\}) = 1 - \overline{\nu}(X)$ for all $\overline{\nu} \in \mathfrak{M}^{\leq 1}(X;\tau)$ we have that

$$\begin{aligned}
\overline{\mu}^1(G) &= \overline{\mu}^1(H) + \overline{\mu}^1(\{\perp\}) \\
&= \overline{\mu}^1(H) + 1 - \lim_j \overline{\mu}_j(X) \\
&= \overline{\mu}^1(H) + \lim_j \overline{\mu}_j^1(\{\perp\}) \\
&= \overline{\mu}^1(H) + \liminf_j \overline{\mu}_j^1(\{\perp\}) \\
&\leq \liminf_j (\overline{\mu}_j^1(H) + \overline{\mu}_j^1(\{\perp\})) \\
&= \liminf_j \overline{\mu}_j^1(G).
\end{aligned}$$

Clearly $\overline{\mu}^1(X) = 1 = \lim_j \overline{\mu}_j^1(X)$. Again by Proposition 7.2 we conclude that $\overline{\mu}_j^1 \to_w \overline{\mu}^1$.

$\boxed{\Leftarrow}$ Clearly $\overline{\mu}(G) \leq \liminf_j \overline{\mu}_j(G)$ since $\overline{\nu}^1$ extends $\overline{\nu}$ for all $\overline{\nu} \in \mathfrak{M}^{\leq 1}(X;\tau)$. In particular the last inequality is true for X; on the other hand X is a clopen subset of $X_\perp$, so we also have that $\limsup_j \overline{\mu}_j(X) \leq \overline{\mu}(X)$ and it follows that $\overline{\mu}(X) = \lim_j \overline{\mu}_j(X)$.

For verifying the last assertion we just have to notice that the map $(\_)^1 : \mathfrak{M}^{\leq 1}(X;\tau) \to \mathfrak{M}^1(X_\perp;\tau)$ is clearly injective. ■

7.5 Generalised splittings and stochastic order

The main objective of this section is to characterise the normalised probabilistic power domain of a stably locally compact space as a stochastically ordered space. When the probabilistic power domain was introduced by Jones and Plotkin [Jon89][JP89] the order among simple valuations was characterised using a 'splitting lemma'[2]. Recall that if $(X, \mathcal{G})$ is a topological space and $\nu, \mu \in \mathbf{P}^1 X$, then $\nu \sqsubseteq \mu$ if and only if $\nu(G) \leq \mu(G)$ for all $G \in \mathcal{G}$. For normalised valuations the splitting lemma states the following.

Proposition 7.12 *Let $(D, \sqsubseteq)$ be a dcpo and $\nu_1 = \sum_{b\in|\nu_1|} r_b\delta_b$, $\nu_2 = \sum_{c\in|\nu_2|} s_c\delta_c$ be elements of $\mathbf{P}^1 D$. Then $\nu_1 \sqsubseteq \nu_2$ if and only if there exists a function $f : |\nu_1| \times |\nu_2| \to [0,1]$ such that for all $b \in |\nu_1|$ and $c \in |\nu_2|$*

1. $\sum_{c\in|\nu_2|} f(b,c) = r_b$.
2. $\sum_{b\in|\nu_1|} f(b,c) = s_c$.
3. $f(b,c) > 0$ *implies* $b \sqsubseteq c$.

Since the simple valuations ν_1 and ν_2 extend uniquely to probability measures $\overline{\nu}_1$ and $\overline{\nu}_2$, this is equivalent to saying that there exists a probability measure $\mathbb{P}$ on $(D \times D, \mathcal{B}D \otimes \mathcal{B}D)$ with marginals $\overline{\nu}_1$ and $\overline{\nu}_2$ and such that $\mathbb{P}(R_\sqsubseteq) = 1$ where $R_\sqsubseteq = \{(x,y) \mid x \sqsubseteq y\}$. Recall that for every probability measure $\mathbb{P}$ on $(X_1 \times X_2, \Sigma_1 \otimes \Sigma_2)$ the marginal $\pi_i[\mathbb{P}]$ of $\mathbb{P}$ on (X_i, Σ_i) is

[2] Not to be confused with the splitting lemma in measure theory as defined in [Top78].

defined as $\pi_i[\mathbb{P}](E) = \mathbb{P}(\pi_i^{-1}E)$ for $i = 1, 2$. The above result was shown by Preston [Pre74, Proposition 2] using the Max-flow, Min-cut theorem, which is the same argument used by Jones in [Jon89, Section 4.4]. As Preston observes the result seems to be folklore in probability theory but is difficult to know where it first appeared. Strassen [Str65, before Theorem 6] quotes a paper by Dall'Aglio[3] published in 1961. Other variants of this result can be found in [Eda95b, Propositions 2.3, 2.5, 3.1 and 3.5] and [Hec96, Lemma 2.7]. In general the problem of finding distributions with given marginals can be stated as follows.

Problem: Given $(X_1, \Sigma_1, \mathbb{P}_1)$, $(X_2, \Sigma_2, \mathbb{P}_2)$ probability spaces and $R \in \Sigma_1 \otimes \Sigma_2$. Can we find a probability $\mathbb{P}$ on $(X_1 \times X_2, \Sigma_1 \otimes \Sigma_2)$ such that $\pi_i[\mathbb{P}] = \mathbb{P}_i$ and $\mathbb{P}(R) = 1$?

The reader should realise the importance of this problem in the light of Kolmogorov's Theorem (Proposition 1.20); since for a family of probability spaces $(X_j, \Sigma_j, \mathbb{P}_j)_{j \in J}$ this would be the first step towards constructing a consistent family of measures on finite subfamilies. Strassen [Str65] studied this problem in Polish spaces and gave necessary and sufficient conditions when R is a closed subset of $X_1 \times X_2$. Since then, the problem has been object of extensive research and a number of generalisations have been given (see [Kaw94], [Edw78], [Lig85, Chapter 2] for references). In particular Dudley's version (see Proposition 7.34 below) uses Philip Hall's marriage lemma [Hal35] which suggests a link with Heckmann's version of the splitting lemma [Hec96, Lemma 2.7]. Sünderhauf has also shown a generalisation of the splitting lemma but the approach is quite different (see Section 7.7.3 below).

For a pospace $(X, \mathcal{G}, \leq)$ the important question becomes to determine if for two probability measures $\mathbb{P}_1$ and $\mathbb{P}_2$ on $(X, \mathcal{B}X)$, there exists a probability measure $\mathbb{P}$ on $(X \times X, \mathcal{B}(X \times X))$ with marginals $\mathbb{P}_1$ and $\mathbb{P}_2$ such that $\mathbb{P}(R_{\leq}) = 1$. In many cases this happens to be equivalent to $\mathbb{P}_1$ being **stochastically smaller** that $\mathbb{P}_2$. That is $\int f d\mathbb{P}_1 \leq \int f d\mathbb{P}_2$ for all bounded increasing continuous functions $f : X \to \mathbb{R}$. We denote this as $\mathbb{P}_1 \prec \mathbb{P}_2$ (see [KKO77][KK78]). Since this equivalence holds in particular for compact Hausdorff spaces, in the light of Proposition 1.13 we can obtain equivalent characterisations for stably compact spaces. We start by recalling some definitions and auxiliary results.

[3] The author has not been able to obtain this paper.

Let $(X, \mathcal{G}, \leq)$ be a pospace. X is a **completely regular ordered space** (CRO-space) or **uniformizable pospace** [Nac65, Chapter II, Section 1] if

1. For all $x \in X$ and $G \in \mathcal{N}(x)$, there exist two continuous functions $f, g : X \to [0,1]$, such that f is increasing, g is decreasing, $f(x) = g(x) = 1$ and $\min(f, g)$ is identically zero in G^{c}.
2. If $x, y \in X$ and $x \not\leq y$, then there exists a continuous increasing function $f : X \to \mathbb{R}$ such that $f(x) > f(y)$.

Proposition 7.13 *(Nachbin) Every compact pospace is a CRO-space. Every subspace of a CRO-space is a CRO-space.*

Proof See Theorem 7 in Chapter II of [Nac65]. ∎

In fact, it can be shown that every CRO-space arises as a subspace of a compact pospace [Law91, Theorem 5].

Corollary 7.14 *Let $(X, \mathcal{G})$ be a locally compact sober space and $\sqsubseteq_s$ its specialisation order in $\mathcal{G}$. Then $(X, \mathcal{G}_p, \sqsubseteq_s)$ is a CRO-space. In particular, every continuous dcpo with the Lawson topology is a CRO-space.*

Proof This is immediate from the previous proposition and Propositions 1.13 and 5.18. ∎

It is important to notice that, contrary to what happens with compact pospaces, there exist examples of completely regular pospaces which are not CRO-spaces [Edw78, Section 7]. That is why the proof of the previous corollary relies on the fact that $(X, \mathcal{G}_p, \sqsubseteq_s)$ is a subspace of a compact pospace. For a topological space $(X, \mathcal{G})$ partially ordered with $\sqsubseteq$, let $\uparrow\lceil X \to \mathbb{R}^\lambda\rceil$ denote the space of increasing bounded continuous real functions. That is,

$$\uparrow\lceil X \to \mathbb{R}^\lambda\rceil := \{f \in \lceil X \to \mathbb{R}^\lambda\rceil \mid x \sqsubseteq y \Rightarrow f(x) \leq f(y) \text{ for all } x, y \in X\}$$

The following result is due to D. A. Edwards.

Proposition 7.15 *[Edw78, Theorems 6.1, 7.1] Let $(X, \mathcal{H}, \leq)$ be a CRO-space and let $\overline{\nu}$ and $\overline{\mu}$ be Radon probability measures on $(X, \mathcal{B}X)$. Then the following conditions are equivalent*

1. *$\overline{\nu} \prec \overline{\mu}$. That is, $\int f d\overline{\nu} \leq \int f d\overline{\mu}$ for all $f \in \uparrow\lceil X \to \mathbb{R}^\lambda\rceil$.*
2. *There exists a Radon probability measure $\mathbb{P}$ on $(X \times X, \mathcal{B}(X \times X))$ with $\pi_1[\mathbb{P}] = \overline{\nu}$, $\pi_2[\mathbb{P}] = \overline{\mu}$ and $\mathbb{P}(R_{\leq}) = 1$.*
3. *$\overline{\nu}(G) \leq \overline{\mu}(G)$ for all upper open $G \subseteq X$. That is, for all $G \in \mathcal{U}_{\leq}$.*

4. *$\overline{\nu}(F) \leq \overline{\mu}(F)$ for all upper closed $F \subseteq X$. That is, $\overline{\mu}(H) \leq \overline{\nu}(H)$ for all $H \in \mathcal{L}_{\leq}$.*

5. *$\int f d\overline{\nu} \leq \int f d\overline{\mu}$ for all increasing bounded LSC functions $f : X \to \mathbb{R}$. That is, $f \in \uparrow\lceil X \to \mathbb{R}^{\leq} \rceil$*

6. *$\int g d\overline{\nu} \leq \int g d\overline{\mu}$ for all increasing bounded USC functions $g : X \to \mathbb{R}$. That is, $g \in \uparrow\lceil X \to \mathbb{R}^{\geq} \rceil$*

Compare also with Corollary 7 in [Ska93]. Now we can state the main results of the section. For a continuous valuation $\lambda \in \mathbf{P}^1(X \times X)$ we define its marginals $\pi_i[\lambda] \in \mathbf{P}^1X$ obviously as $\pi_i[\lambda](G) = \lambda(\pi_i^{-1}G)$ for $i = 1, 2$.

Theorem 7.16 ***Probabilistic powerdomains as stochastically ordered spaces.*** *Let $(X, \mathcal{G})$ be a stably compact space. Let ν, μ be elements of $\mathbf{P}^1X$ and $\overline{\nu}$, $\overline{\mu}$ denote their unique extensions to Radon measures on $(X, \sigma(\mathcal{G}_p))$ respectively. Then*

$$\nu \sqsubseteq \mu \text{ if and only if } \overline{\nu} \prec \overline{\mu}.$$

In which case there exists a continuous valuation $\lambda \in \mathbf{P}^1(X \times X)$ with $\pi_1[\lambda] = \nu$, $\pi_2[\lambda] = \mu$ and $\overline{\lambda}(\{(x, y) \mid x \sqsubseteq_s y\}) = 1$.

Proof By Proposition 1.13 we know that $(X, \mathcal{G}_p, \sqsubseteq_s)$ is a compact pospace, hence a CRO-space, where the upper open sets are precisely the sets in $\mathcal{G}$. The first claim follows form the equivalence of (1) $\Leftrightarrow$ (3) in Proposition 7.15. The second claim follows from the condition (2) in the same proposition since Radon measures are τ-smooth, hence their restrictions are continuous valuations. ■

Corollary 7.17 *Let $(X, \mathcal{G})$ be a stably locally compact space. Let ν, μ be elements of $\mathbf{P}^1X$ and $\overline{\nu}$, $\overline{\mu}$ denote their unique extensions to Radon measures on $(X, \sigma(\mathcal{G}_p))$ respectively. Then*

$$\nu \sqsubseteq \mu \text{ if and only if } \overline{\nu} \prec \overline{\mu}.$$

In which case there exists a continuous valuation $\lambda \in \mathbf{P}^1(X \times X)$ with $\pi_1[\lambda] = \nu$, $\pi_2[\lambda] = \mu$ and $\overline{\lambda}(\{(x, y) \mid x \sqsubseteq_s y\}) = 1$.

Proof Recall the constructions in Section 7.4. Apply the previous theorem to $\nu^\perp$ and $\mu^\perp$ in $\mathbf{P}^1X_\perp$ and let $\mathbb{P}$ be the probability measure given by part (2) of Proposition 7.15. Notice that

$$\mathbb{P}(\{\perp\} \times X_\perp) = \overline{\nu}^\perp(\{\perp\}) = \nu^\perp(X_\perp) - \nu^\perp(X) = 0.$$

Similarly $\mathbb{P}(X_\perp \times \{\perp\}) = 0$. Therefore $\mathbb{P}(X_\perp \times X_\perp) = \mathbb{P}(X \times X)$ and $\mathbb{P}$ restricts to a probability measure on $X \times X$ with marginals $\overline{\nu}$ and $\overline{\mu}$. ■

Remark 7.18 *If $(X, \mathcal{G})$ is a locally compact sober space, then we know that $(X, \mathcal{G}_p, \sqsubseteq_s)$ is a CRO-space. Nevertheless for a valuation $\nu \in \mathbf{P}^1X$ we only know that ν extends to a probability measure on $\mathcal{B}X$ but not necessarily on $\mathcal{B}_pX = \sigma(\mathcal{G}_p)$.*

If $(X, \mathcal{G})$ is a second countable locally compact sober space, then $(X, \mathcal{G}_p)$ is a Polish space (Proposition 3.10-(5)) and $\sigma(\mathcal{G}) = \sigma(\mathcal{G}_p)$. We know that in a Polish space all probability measures are Radon (Proposition 1.28) therefore we have the following:

Proposition 7.19 *[Nor92, Theorem 4.16] Let $(X, \mathcal{G})$ be a second countable and locally compact sober space. Let ν, μ be elements of $\mathbf{P}^1X$ and $\overline{\nu}$, $\overline{\mu}$ denote their unique extensions to Radon measures on $(X, \sigma(\mathcal{G}_p))$ respectively. Then $\nu \sqsubseteq \mu$ if and only if $\overline{\nu} \prec \overline{\mu}$.*

7.6 Stable compactness

Jung and Tix showed the following result and posed the open question whether the same would hold for stably compact spaces.

Proposition 7.20 *[JT98, Theorem 4.2] Suppose that $(D, \sqsubseteq)$ is a stably compact domain with least element. Then $(\mathbf{P}D, \sqsubseteq)$ and $(\mathbf{P}^1D, \sqsubseteq)$ are stably compact domains (with least element).*

We show that if $(X, \mathcal{G})$ is a stably compact space, then $(\mathbf{P}^1X, \mathcal{G}_\sigma(\mathbf{P}^1X))$ and $(\mathbf{P}X, \mathcal{G}_\sigma(\mathbf{P}X))$ are stably compact spaces. We will need the following result by Kawabe.

Proposition 7.21 *(Kawabe) Let $(X_1, \mathcal{G}_1)$ and $(X_2, \mathcal{G}_2)$ be regular Hausdorff spaces (T_3). Suppose that $(\overline{\lambda}_j)_{j\in J}$ is a net in $\mathfrak{M}_+(X_1 \times X_2)$ such that*

- $\limsup_j \overline{\lambda}_j(X_1 \times X_2) < \infty$
- $\pi_1[\overline{\lambda}_j] \to_w \overline{\mu} \in \mathfrak{M}_+(X_1; \tau)$
- $\pi_2[\overline{\lambda}_j] \to_w \overline{\nu} \in \mathfrak{M}_+(X_2; \tau)$.

Then every subnet of $(\overline{\lambda}_j)_{j\in J}$ has a further subnet converging weakly to a measure $\overline{\lambda} \in \mathfrak{M}_+(X_1 \times X_2; \tau)$ such that $\pi_1[\overline{\lambda}] = \overline{\mu}$ and $\pi_2[\overline{\lambda}] = \overline{\nu}$.

Proof See Theorem 1 in [Kaw94]. ■

The previous proposition enables us to prove the following important theorem (which is probably known but we could not find a reference).

Theorem 7.22 *Suppose that $(X, \mathcal{H}, \leq)$ is a compact pospace (hence a Hausdorff space). Then $(\mathfrak{M}^1(X;\tau), \mathcal{G}_w(\mathfrak{M}X), \prec)$ is a compact pospace.*

Proof By part (9) of Proposition 7.3 we know that $(\mathfrak{M}^1(X;\tau), \mathcal{G}_w(\mathfrak{M}X))$ is a compact Hausdorff space; it remains to check that $R^1_\prec = \{(\overline{\mu}, \overline{\nu}) \mid \overline{\mu} \prec \overline{\nu}\}$ is a closed subset of $\mathfrak{M}^1(X;\tau) \times \mathfrak{M}^1(X;\tau)$. Suppose that $(\overline{\mu}_j, \overline{\nu}_j)_{j \in J}$ is a net in $R^1_\prec$ such that $(\overline{\mu}_j, \overline{\nu}_j) \to (\overline{\mu}, \overline{\nu}) \in \mathfrak{M}^1(X;\tau) \times \mathfrak{M}^1(X;\tau)$. Then $(\overline{\mu}_j)_{j \in J}$ and $(\overline{\nu}_j)_{j \in J}$ are nets in $\mathfrak{M}^{\leq 1}(X;\tau)$ such that $(\overline{\mu}_j) \to_w \overline{\mu}$ and $(\overline{\nu}_j) \to_w \overline{\nu}$. Since $\overline{\mu}_j \prec \overline{\nu}_j$ then by part (2) of Proposition 7.15, there exists a $\overline{\lambda}_j \in \mathfrak{M}^1(X \times X;\tau)$ with $\pi_1[\overline{\lambda}_j] = \overline{\mu}_j$, $\pi_2[\overline{\lambda}_j] = \overline{\nu}_j$ and $\overline{\lambda}_j(R_\leq) = 1$ for all $j \in J$. Clearly $\pi_1[\overline{\lambda}_j] \to_w \overline{\mu}$ and $\pi_2[\overline{\lambda}_j] \to_w \overline{\nu}$, therefore by Proposition 7.21 there exists a subnet $(\overline{\lambda}_k)_{k \in I}$ such that $\overline{\lambda}_k \to_w \overline{\lambda} \in \mathfrak{M}_+(X \times X;\tau)$ with $\pi_1[\overline{\lambda}] = \overline{\mu}$ and $\pi_2[\overline{\lambda}] = \overline{\nu}$.

Notice that $\overline{\lambda} \in \mathfrak{M}^1(X \times X;\tau)$ since $\overline{\lambda}(X \times X) = \pi_1[\overline{\lambda}](X) = \overline{\mu}(X) = 1$. Since $(X, \mathcal{H}, \leq)$ is a compact pospace, we know that $R_\leq$ is a closed and compact subset of $X \times X$. Hence by condition (5) of Proposition 7.2 we have that $\overline{\lambda}(R_\leq) \geq \limsup_k \overline{\lambda}_k(R_\leq) = 1$. Since $\pi_1[\overline{\lambda}] = \overline{\mu}$ and $\pi_2[\overline{\lambda}] = \overline{\nu}$ by the equivalence (1) $\Leftrightarrow$ (2) in Proposition 7.15 we obtain that $\overline{\mu} \prec \overline{\nu}$. Hence $(\overline{\mu}, \overline{\nu}) \in R^1_\prec$ and we have shown that $R^1_\prec$ is closed. ■

Although the stochastic order $\prec$ was only defined for probability measures, we can extend it to $(\mathfrak{M}^{\leq 1}(X;\tau), \mathcal{G}_w(\mathfrak{M}X))$ as follows. Suppose that $(X, \mathcal{H}, \leq)$ is a compact pospace and let $\mathcal{G} = \{G \in \mathcal{H} \mid G = \uparrow G\}$. For $\overline{\mu}$, $\overline{\nu} \in \mathfrak{M}^{\leq 1}(X;\tau)$, we define

$$\overline{\mu} \sqsubseteq \overline{\nu} \text{ if and only if } \overline{\mu}(G) \leq \overline{\nu}(G) \text{ for all } G \in \mathcal{G}.$$

By Proposition 1.13 we know that $(X, \mathcal{G})$ is a stably compact space where the specialisation order agrees with $\leq$. So the previous definition could be rephrased as:

$$\overline{\mu} \sqsubseteq \overline{\nu} \text{ if and only if } \mu \restriction_{\mathcal{G}} \sqsubseteq \nu \restriction_{\mathcal{G}}.$$

Theorem 7.23 *Suppose that $(X, \mathcal{H}, \leq)$ is a compact pospace. Then $(\mathfrak{M}^{\leq 1}(X;\tau), \mathcal{G}_w(\mathfrak{M}X), \sqsubseteq)$ is a compact pospace.*

Proof The argument is similar to the one in the proof of Theorem 7.22. The main difference is that we have to embed $\mathfrak{M}^{\leq 1}(X;\tau)$ into $\mathfrak{M}^1(X_\perp;\tau)$ via Lemma 7.11 in order to apply the latter theorem. By Corollary 7.6 we know that $(\mathfrak{M}^{\leq 1}(X;\tau), \mathcal{G}_w(\mathfrak{M}X))$ is a compact Hausdorff space. We verify that

$$R_{\sqsubseteq} = \{(\overline{\mu},\overline{\nu}) \mid \mu \restriction_{\mathcal{G}} \sqsubseteq \nu \restriction_{\mathcal{G}}\}$$

is a compact subset of $\mathfrak{M}^{\leq 1}(X;\tau) \times \mathfrak{M}^{\leq 1}(X;\tau)$. Suppose that $(\overline{\mu}_j,\overline{\nu}_j)_{j\in J}$ is a net in $R_{\sqsubseteq}$. Since $\mathfrak{M}^{\leq 1}(X;\tau) \times \mathfrak{M}^{\leq 1}(X;\tau)$ is compact (endowed with the product topology of the weak topologies) then $(\overline{\mu}_j,\overline{\nu}_j)_{j\in J}$ has a subnet $(\overline{\mu}_i,\overline{\nu}_i)_{i\in I}$ such that $(\overline{\mu}_i,\overline{\nu}_i) \to (\overline{\mu},\overline{\nu})$ for some $(\overline{\mu},\overline{\nu}) \in \mathfrak{M}^{\leq 1}(X;\tau) \times \mathfrak{M}^{\leq 1}(X;\tau)$. Then $(\overline{\mu}_i)_{i\in I}$ and $(\overline{\nu}_i)_{i\in I}$ are nets in $\mathfrak{M}^{\leq 1}(X;\tau)$ such that $(\overline{\mu}_i) \to_w \overline{\mu}$ and $(\overline{\nu}_i) \to_w \overline{\nu}$. By Lemma 7.11 we have that $(\overline{\mu}_i^1,\overline{\nu}_i^1) \to (\overline{\mu}^1,\overline{\nu}^1)$ in $\mathfrak{M}^1(X_\perp;\tau)\times\mathfrak{M}^1(X_\perp;\tau)$. Furthermore, $\overline{\mu}_i^1 \prec \overline{\nu}_i^1$ for all $i \in I$. By Theorem 7.22 we know that $(\mathfrak{M}^1(X_\perp;\tau),\mathcal{G}_w(\mathfrak{M}X),\prec)$ is a compact pospace, hence $R^1_{\prec}$ is closed and it follows that $\overline{\mu}^1 \prec \overline{\nu}^1$. By the equivalence (1) $\Leftrightarrow$ (3) in Proposition 7.15 we have that

$$\overline{\mu}^1 \prec \overline{\nu}^1 \text{ if and only if } \overline{\mu}^1(G) \leq \overline{\nu}^1(G) \text{ for all } \textbf{upper} \text{ open sets } G \in \mathcal{H}_\perp.$$

In particular $\overline{\mu}(G) = \overline{\mu}^1(G) \leq \overline{\nu}^1(G) = \overline{\nu}(G)$ for all upper open sets $G \in \mathcal{H}$. Hence $(\overline{\mu},\overline{\nu}) \in R_{\sqsubseteq}$ and we conclude that $R_{\sqsubseteq}$ is compact and therefore closed (since $(\mathfrak{M}^{\leq 1}(X;\tau),\mathcal{G}_w(\mathfrak{M}X))$ is Hausdorff). ■

In fact, the convergence in the weak topology ensures that we can strengthen the order $\sqsubseteq$ on $(\mathfrak{M}^{\leq 1}(X;\tau),\mathcal{G}_w(\mathfrak{M}X))$ and still obtain a compact pospace: for $\overline{\mu},\overline{\nu} \in \mathfrak{M}^{\leq 1}(X;\tau)$ define,

$$\overline{\mu} \trianglelefteq \overline{\nu} \text{ if and only if } \overline{\mu} \sqsubseteq \overline{\nu} \text{ and } \overline{\mu}(X) = \overline{\nu}(X).$$

The same argument as above shows that

Theorem 7.24 *Suppose that $(X,\mathcal{H},\leq)$ is a compact pospace. Then $(\mathfrak{M}^{\leq 1}(X;\tau),\mathcal{G}_w(\mathfrak{M}X),\trianglelefteq)$ is a compact pospace.*

Recall that,

$$\begin{aligned}
\mathcal{U}_{\sqsubseteq} &\equiv \{G \in \mathcal{G}_w(\mathfrak{M}X) \mid G \text{ is upper wrt } \sqsubseteq\}\\
\mathcal{L}_{\sqsubseteq} &\equiv \{H \in \mathcal{G}_w(\mathfrak{M}X) \mid H \text{ is lower wrt } \sqsubseteq\}\\
\mathcal{U}_{\trianglelefteq} &\equiv \{G \in \mathcal{G}_w(\mathfrak{M}X) \mid G \text{ is upper wrt } \trianglelefteq\}\\
\mathcal{L}_{\trianglelefteq} &\equiv \{H \in \mathcal{G}_w(\mathfrak{M}X) \mid H \text{ is lower wrt } \trianglelefteq\}
\end{aligned}$$

then by Proposition 1.13 we have that $(\mathfrak{M}^{\leq 1}(X;\tau),\mathcal{U}_{\sqsubseteq})$ and $(\mathfrak{M}^{\leq 1}(X;\tau),\mathcal{U}_{\trianglelefteq})$ are stably compact spaces whose patch space is $(\mathfrak{M}^{\leq 1}(X;\tau),\mathcal{G}_w(\mathfrak{M}X))$. Notice that if $G \in \mathcal{G}_w(\mathfrak{M}X)$ is $\sqsubseteq$-upper, then G is $\trianglelefteq$-upper, and if $H \in \mathcal{G}_w(\mathfrak{M}X)$ is $\sqsubseteq$-lower, then H is $\trianglelefteq$-lower. Hence, $\mathcal{U}_{\sqsubseteq} \subseteq \mathcal{U}_{\trianglelefteq}$ and $\mathcal{L}_{\sqsubseteq} \subseteq \mathcal{L}_{\trianglelefteq}$. Strangely enough we have that $\mathcal{G}_w(\mathfrak{M}X) = \mathcal{U}_{\trianglelefteq} \vee \mathcal{L}_{\trianglelefteq} = \mathcal{U}_{\sqsubseteq} \vee \mathcal{L}_{\sqsubseteq}$. This situation is more easily understood through the following simpler example.

Example 7.25 *Let $Y = \{1, 2, 3\}$ and $\mathcal{H}$ be the discrete topology on Y. Let $\sqsubseteq$ be the minimum partial order such that $1 \sqsubseteq 2 \sqsubseteq 3$; and $\trianglelefteq$ be the minimum partial order on Y such that $1 \trianglelefteq 2$. Then*

$$\begin{aligned} R_{\sqsubseteq} &= \{(1,1),(1,2),(2,2),(2,3),\quad (3,3)\} \\ R_{\trianglelefteq} &= \{(1,1),(1,2),(2,2),\qquad\qquad (3,3)\} \end{aligned}$$

Since the product topology of discrete spaces is also discrete, then $(Y, \mathcal{H}, \sqsubseteq)$ and $(Y, \mathcal{H}, \trianglelefteq)$ are clearly compact pospaces. Now observe that

$$\begin{aligned} \mathcal{U}_{\sqsubseteq} &= \{\emptyset, Y, \qquad\qquad\qquad \{2,3\},\{3\}\} \\ \mathcal{L}_{\sqsubseteq} &= \{\emptyset, Y, \qquad\qquad\qquad \{1\},\{1,2\}\} \\ \mathcal{U}_{\trianglelefteq} &= \{\emptyset, Y, \{1,2\},\{2\}, \quad \{2,3\},\{3\}\} \\ \mathcal{L}_{\trianglelefteq} &= \{\emptyset, Y, \{3\},\{1,3\}, \quad \{1\},\{1,2\}\} \end{aligned}$$

and it is easy to see that $\mathcal{H} = \mathcal{U}_{\trianglelefteq} \vee \mathcal{L}_{\trianglelefteq} = \mathcal{U}_{\sqsubseteq} \vee \mathcal{L}_{\sqsubseteq}$.

The main result of the section is the following.

Theorem 7.26 ***Stable compactness of normalised probabilistic powerdomains of stably compact spaces.*** *Suppose that $(X, \mathcal{G})$ is a stably compact space. Then $\mathbf{P}^1X$ can be given a topology which makes it a stably compact space; this topology is finer than the pointwise convergence topology $\mathcal{G}_{pw}(\mathbf{P}^1X)$ and coarser than the Scott topology $\mathcal{G}_{\sigma}(\mathbf{P}^1X)$.*

Proof Let $\leq$ denote the specialisation order in $(X, \mathcal{G})$. By Proposition 1.13 we know that $(X, \mathcal{G}_p, \leq)$ is a compact pospace. Recall that $X^p \equiv (X, \mathcal{G}_p)$. Then by Theorem 7.22 we have that $(\mathfrak{M}^1(X^p; \tau), \mathcal{G}_w(\mathfrak{M}X^p), \prec)$ is a compact pospace. Consider $\mathcal{U}_{\prec} \equiv \{U \in \mathcal{G}_w(\mathfrak{M}X^p) \mid U = \uparrow_{\prec} U\}$: the set of $\prec$-upper sets in $\mathcal{G}_w(\mathfrak{M}X^p)$. Again by Proposition 1.13 we have that $(\mathfrak{M}^1(X^p; \tau), \mathcal{U}_{\prec})$ is a stably compact space such that

- the specialisation order of $\mathcal{U}_{\prec}$ is $\prec$,
- the patch topology $(\mathcal{U}_{\prec})_p$ agrees with $\mathcal{G}_w(\mathfrak{M}X^p)$,
- the cocompact topology $(\mathcal{U}_{\prec})_{cc}$ is the paving $\mathcal{L}_{\prec}$ of $\prec$-lower open sets in $\mathcal{G}_w(\mathfrak{M}X^p)$.

In the following, $(\overline{\mu}_j)_{j \in J}$ is any net in $\mathfrak{M}^1(X^p; \tau)$ and $\overline{\mu} \in \mathfrak{M}^1(X^p; \tau)$.

1. **Claim:** $\mathbf{P}^1X$ and $\mathfrak{M}^1(X^p; \tau)$ are in one-to-one correspondence. By Proposition 3.37 we have that every continuous valuation $\nu \in \mathbf{P}^1X$ extends uniquely to a Radon measure $\overline{\nu} \in \mathfrak{M}^1(X^p; R)$. Since X^p is a compact Hausdorff space then $\mathfrak{M}^1(X^p; \tau) = \mathfrak{M}^1(X^p; R)$. Conversely

every $\overline{\mu} \in \mathfrak{M}^1(X^p;\tau)$ restricts to a continuous valuation $\mu \in \mathbf{P}^1 X^p$, which in turn restricts to a continuous valuation in $\mathbf{P}^1 X$ since $\mathcal{G} \subseteq \mathcal{G}_p$.

2. **Claim:** $(\mathbf{P}^1 X, \sqsubseteq)$ and $(\mathfrak{M}^1(X^p;\tau), \prec)$ are isomorphic as posets. This follows from Theorem 7.16 since $\nu \sqsubseteq \mu$ if and only if $\overline{\nu} \prec \overline{\mu}$.

3. **Claim:** if $\overline{\mu}_j \to_{\mathcal{U}_\prec} \overline{\mu}$, then $\mu_j \to_{pw} \mu$. In order to derive a contradiction, suppose that $(\mu_j)_{j\in J}$ does not converge pointwise to μ. Then (by Remark 6.18-(3)) there exists $G_0 \in \mathcal{G}$ such that $\liminf_j \mu_j(G_0) < \mu(G_0)$. Fix $M \in \mathbb{R}$ with $\liminf_j \mu_j(G_0) < M < \mu(G_0)$. Then by definition of lim inf we have that $\inf\{\mu_j(G_0) \mid m \leq j\} < M$ for all $m \in J$. Hence for every $m \in J$ there exists $j_m \in J$ such that $m \leq j_m$ and

$$\mu_{j_m}(G_0) < M.$$

Since $(\mathfrak{M}^1(X^p;\tau), \mathcal{G}_w(\mathfrak{M}X^p))$ is a compact Hausdorff space, $(\overline{\mu}_{j_m})_{m\in J}$ has a further subnet $(\overline{\mu}_i)_{i\in I}$ converging weakly to $\overline{\nu} \in \mathfrak{M}^1(X^p;\tau)$. Therefore by condition (iv) in Proposition 7.2 we have that

$$\overline{\nu}(G_0) \leq \liminf_i \overline{\mu}_i(G_0) \leq M < \overline{\mu}(G_0).$$

This means that $\overline{\mu} \not\prec \overline{\nu}$. That is, $\overline{\mu} \notin cl_{\mathcal{U}_\prec}(\overline{\nu})$ since $\prec$ is the specialisation order in $\mathcal{U}_\prec$. Since $(\mathfrak{M}^1(X^p;\tau), \mathcal{G}_w(\mathfrak{M}X^p), \prec)$ is a compact pospace, then $(\mathfrak{M}^1(X^p;\tau), \mathcal{U}_\prec, \mathcal{L}_\prec)$ is a pseudo-Hausdorff bitopological space (see Section 3.7). Hence, there exists an upper open set $U \in \mathcal{U}_\prec$ and an open lower set $L \in \mathcal{L}_\prec$, such that $\overline{\mu} \in U$, $\overline{\nu} \in L$ and $U \cap L = \emptyset$. Since $\overline{\mu}_i \to_w \overline{\nu}$, then $(\overline{\mu}_i)_{i\in I}$ is in L eventually.

But since $\overline{\mu}_j \to_{\mathcal{U}_\prec} \overline{\mu}$ and $(\overline{\mu}_i)_{i\in I}$ is a subnet of $(\overline{\mu}_j)_{j\in J}$, then $\overline{\mu}_i \to_{\mathcal{U}_\prec} \overline{\mu}$. Which means that $(\overline{\mu}_i)_{i\in I}$ is in U eventually; a contradiction since $U \cap L = \emptyset$.

By Claim (2) we can think of $(\mathbf{P}^1 X, \sqsubseteq)$ as a subset of $(\mathfrak{M}^1(X^p;\tau), \prec)$. Thus, Claim (3) is equivalent to saying that the subspace topology $\mathcal{U}_\prec \restriction_{\mathbf{P}}$ induced by $\mathcal{U}_\prec$ on $\mathbf{P}^1 X$ is finer than $\mathcal{G}_{pw}(\mathbf{P}^1 X)$. On the other hand, $(\mathfrak{M}^1(X^p;\tau), \mathcal{U}_\prec)$ being a stably compact space is in particular a sober space. Then by part (1) of Proposition 3.9 we have that $\mathcal{U}_\prec$ is coarser than the Scott topology on $(\mathfrak{M}^1(X^p;\tau), \prec)$. But the topology on $\mathbf{P}^1 X$ induced by the Scott topology of $(\mathfrak{M}^1(X^p;\tau), \prec)$ is precisely the Scott topology $\mathcal{G}_\sigma(\mathbf{P}^1 X)$ of $(\mathbf{P}^1 X, \sqsubseteq)$ (since they are isomorphic posets). Summarising, we have shown that,

$$\mathcal{G}_{pw}(\mathbf{P}^1 X) \subseteq \mathcal{U}_\prec \restriction_{\mathbf{P}} \subseteq \mathcal{G}_\sigma(\mathbf{P}^1 X). \blacksquare$$

From Theorem 7.23, repeating the argument in the proof of Theorem 7.26, we obtain the corresponding result for the probabilistic powerdomain.

Theorem 7.27 *Suppose that* $(X, \mathcal{G})$ *is a stably compact space. Then* $\mathbf{P}X$ *can be given a topology which makes it a stably compact space; this topology is finer than the pointwise convergence topology* $\mathcal{G}_{pw}(\mathbf{P}X)$ *and coarser than the Scott topology* $\mathcal{G}_\sigma(\mathbf{P}X)$.

But there is a very important difference with respect to the normalised case. Recall that a net $(\overline{\mu}_j)_{j\in J}$ in $\mathfrak{M}^{\leq 1}(X;\tau)$ converges weakly to $\overline{\mu} \in \mathfrak{M}^{\leq 1}(X;\tau)$ if

$$\overline{\mu}(G) \leq \liminf_j \overline{\mu}_j(G) \text{ for all } G \in \mathcal{G} \text{ and } \overline{\mu}(X) = \lim_j \overline{\mu}_j(X).$$

So is not difficult to see that the specialisation order in $\mathfrak{M}^{\leq 1}(X;\tau)$ corresponding to the weak topology is $\trianglelefteq$ and not $\sqsubseteq$.

7.7 Further remarks on asymmetric topology and metrics on spaces of measures

The objective of this section is to review some of the theory of metrics on spaces of measures, highlight some links with domain theory and suggest some lines for future research. We remark that there are no new results in this section. We start our exposition with the following observation. Suppose that $(X, \mathcal{H}, \leq)$ is a compact pospace, then (by Proposition 1.13) we know that $(X, \mathcal{U}_\leq)$ is a stably compact space such that $\mathcal{L}_\leq = (\mathcal{U}_\leq)_{cc}$ and $\mathcal{H} = \mathcal{U}_\leq \vee \mathcal{L}_\leq$. Furthermore, by part (vii)-4 of Lemma 3.41 we have that $(X, \mathcal{L}_\leq)$ is also a stably compact space. Proposition 7.15 implies that if $\overline{\mu}$ and $\overline{\nu}$ are Radon probability measures on $(X, \mathcal{B}X)$, then the following conditions are equivalent:

- $\overline{\mu} \prec \overline{\nu}$
- $\overline{\mu} \restriction_{\mathcal{U}_\leq} \sqsubseteq \overline{\nu} \restriction_{\mathcal{U}_\leq}$ as valuations on $(X, \mathcal{U}_\leq)$.
- $\overline{\nu} \restriction_{\mathcal{L}_\leq} \sqsubseteq \overline{\mu} \restriction_{\mathcal{L}_\leq}$ as valuations on $(X, \mathcal{L}_\leq)$.

Proposition 3.33 also suggests that what can be said about $(X, \mathcal{H})$ for τ-smooth measures splits into two statements for $(X, \mathcal{U}_\leq)$ and $(X, \mathcal{L}_\leq)$ which are dual of each other. This situation is not really surprising if we remember that the categories **CoPS** and **StCo** are in fact isomorphic [Smy91, Theorem 20]. Recall from Section 3.7 that the minimum requirement for extending a topological notion to a bitopological setting, is that the original notion should be recovered from the bitopological notion applied to $(X, \mathcal{G}, \mathcal{G})$. The following abstract recipe seems to be the underlying idea for many results (and many more to come) in asymmetric topology:

1. Take $(X, \mathcal{G}, \mathcal{G}^*)$ a bitopological space where $\mathcal{G}$ and $\mathcal{G}^*$ are related under some type of duality and such that $(X, \mathcal{G} \vee \mathcal{G}^*)$ is a Hausdorff space.
2. Choose a classic result or definition R for Hausdorff spaces and try to extend it to $(X, \mathcal{G})$ and $(X, \mathcal{G}^*)$.
3. Verify that: R holds for $(X, \mathcal{G})$ and some dual R^* of R holds for $(X, \mathcal{G}^*)$, if and only if, the classical version of R holds for $(X, \mathcal{G} \vee \mathcal{G}^*)$.

Important instances are of course: $(D, \mathcal{G}_\sigma D, \mathcal{G}_l D)$ when D is a continuous dcpo and $(X, \mathcal{G}, \mathcal{G}_{cc})$ when X is a topological space; the setting $(X, \mathcal{U}_\leq, \mathcal{L}_\leq)$ where $(X, \mathcal{H}, \leq)$ is a compact pospace is one of the best possible scenarios.

With the previous information in mind we turn our attention to metric spaces. Given a metric space (X, d) there exist two important metrics which can be defined on $\mathfrak{M}^1(X)$, namely the Lévy-Prohorov metric and the Hutchison metric. Both of them induce the weak topology on $\mathfrak{M}^1(X)$ (see details below). In the context of domain theory, the Hutchinson metric was studied by Edalat in [Eda95d] and [Eda96] (see also [Eda97a] and [EH98]). Norberg has shown the following result. Recall that $X^p \equiv (X, \mathcal{G}_p)$.

Proposition 7.28 *[Nor97, Theorem 6.3] Suppose that $(\Omega, \Gamma, \mathbb{P})$ is a probability space, $(X, \mathcal{G})$ is second countable locally compact sober space, $\xi_n : \Omega \to X$ is a sequence of random functions and $\xi : \Omega \to X$ is a random function. Then $\mathbb{P}_{\xi_n} \to_w \mathbb{P}_\xi$ in the weak topology of $\mathfrak{M}^1(X^p)$ if and only if the following two conditions are satisfied:*

1. $\mathbb{P}_\xi(G) \leq \liminf_n \mathbb{P}_{\xi_n}(G)$ *for all* $G \in \mathcal{G}$.
2. $\limsup_n \mathbb{P}_{\xi_n}(Q) \leq \mathbb{P}_\xi(Q)$ *for all* $Q \in \mathcal{Q}$.

The reader will also find similar results in [Nor97] for continuous dcpo's. If a metric space (X, d) is further endowed with a partial order $\leq$ which makes it a pospace, we believe it should be possible to derive asymmetric formulations of the Lévy-Prohorov and Hutchinson metrics. We remind the reader that a second countable locally compact sober space (in particular an ω-continuous domain) becomes a Polish space if topologised with the Lawson topology (Proposition 3.10-(5)). By "asymmetric formulation of a metric" specifically we have in mind: quasi-metrics, partial metrics and pseudo-metrics. This brings us to the work of Sünderhauf (Section 7.7.3), who defined a quasi-metric compatible with the preorder $\sqsubseteq$ on the space of continuous valuations $\mathbf{V}Y$ of a given topological space $(Y, \mathcal{H})$.

In relation to domain theory the theory of generalised metrics has received a considerable amount of attention, the interested reader can find more information in: [BBR96], [O'N96], [Sch96], [Ver97, Section 8] and [Smy91] for instance.

7.7.1 The Lévy-Prohorov metric

For the rest of this subsection $(X, \mathcal{G})$ will denote a Hausdorff topological space. A subset $\mathcal{P} \subseteq \mathfrak{M}_+(X; R)$ is called **uniformly tight** if for every $\varepsilon > 0$ there exists a compact set $K \in \mathcal{K}$ such that

$$\overline{\mu}(X) - \varepsilon < \overline{\mu}(K) \leq \overline{\mu}(X) \text{ for all } \overline{\mu} \in \mathcal{P}.$$

For any topological space $(Y, \mathcal{H})$, we say that $R \subseteq Y$ is **relatively compact** if $cl(R)$ is compact.

Proposition 7.29 *Suppose that $(X, \mathcal{G})$ is a Hausdorff topological space. Then,*

1. *If $\mathcal{P} \subseteq \mathfrak{M}_+(X; R)$ is uniformly tight and $\sup\{\overline{\mu}(X) \mid \overline{\mu} \in \mathcal{P}\} < \infty$, then $\mathcal{P}$ is relatively compact in $(\mathfrak{M}_+(X; R), \mathcal{G}_w(\mathfrak{M}X))$.*

2. *If $\mathcal{P} \subseteq \mathfrak{M}_+(X; R)$ is relatively compact in $(\mathfrak{M}_+(X; R), \mathcal{G}_w(\mathfrak{M}X))$, then $\sup\{\overline{\mu}(X) \mid \overline{\mu} \in \mathcal{P}\} < \infty$.*

3. *If X is locally compact or Polish, then*

   $\mathcal{P} \subseteq \mathfrak{M}_+(X; R)$ is uniformly tight and $\sup\{\overline{\mu}(X) \mid \overline{\mu} \in \mathcal{P}\} < \infty$ if and only if $\mathcal{P}$ is relatively compact in $(\mathfrak{M}_+(X; R), \mathcal{G}_w(\mathfrak{M}X))$.

Proof See Theorems 3 and 4 in the Appendix of [Sch73] or [Top74]. ■

The following results and definitions are extracted from [Top74]. A regular Hausdorff (T_3) space is called a **Prohorov space** if the relatively compact subsets of $(\mathfrak{M}_+(X; R), \mathcal{G}_w(\mathfrak{M}X))$ are precisely the uniformly tight subsets $\mathcal{P} \subseteq \mathfrak{M}_+(X; R)$ satisfying $\sup\{\overline{\mu}(X) \mid \overline{\mu} \in \mathcal{P}\} < \infty$. Hence part (3) of the last proposition can be rephrased as locally compact Hausdorff spaces and Polish spaces are Prohorov spaces. It can be shown that a T_3 space X is a Prohorov space if and only if every compact subset of $\mathfrak{M}_+(X; R)$ is uniformly tight. Suppose that $(Y, \mathcal{H})$ is any topological space. A net $(x_j)_{j \in J}$ in Y is said to be **compact** if every subnet of $(x_j)_{j \in J}$ has a further subnet which converges to an element of Y. A set $B \subseteq Y$ is said to be **net compact** if every net in B is compact (note that the convergence of the sub-subnet is to a point in Y but not necessarily in B). It should be clear from part (3) of Proposition 1.9 that every relatively compact set is net compact. If Y is T_3, then a set is net compact if and only if it is relatively compact. From the previous remarks and Proposition 7.29 we obtain.

Corollary 7.30 *Suppose that $(X, \mathcal{G})$ is a Prohorov space. Then a family $\mathcal{P} \subseteq \mathfrak{M}^1(X; R)$ is net compact if and only if $\mathcal{P}$ is uniformly tight. In particular, for any net $(\overline{\mu}_j)_{j \in J}$ in $\mathfrak{M}^1(X; R)$ with $\overline{\mu}_j \to \overline{\mu} \in \mathfrak{M}^1(X; R)$ we have that the set $\{\overline{\mu}_j \mid j \in J\}$ is uniformly tight.*

The original version of Prohorov's Theorem [Pro56] follows immediately from the last corollary, but before we can present it, we need to introduce one definition more. Suppose that X is a Hausdorff space and $\mathcal{P} \subseteq \mathfrak{M}^1(X; R)$. We say that $\mathcal{P}$ is **relatively sequentially compact** if every sequence $(\overline{\mu}_n)$ in $\mathcal{P}$ has a convergent subsequence (in the weak topology).

Proposition 7.31 *(Prohorov's Theorem) Suppose that (X, d) is a Polish space, then $\mathcal{P} \subseteq \mathfrak{M}^1(X)$ is relatively sequentially compact if and only if $\mathcal{P}$ is uniformly tight.*

We remind the reader that if X is Polish then, $\mathfrak{M}^1(X) = \mathfrak{M}^1(X; \tau) = \mathfrak{M}^1(X; R) = \mathfrak{M}^1(X; F)$ (Proposition 1.28-(1)). Now compare with the following result by Norberg.

Proposition 7.32 *[Nor97, Theorem 6.7] Suppose that $(\Omega, \Gamma, \mathbb{P})$ is a probability space, $(X, \mathcal{G})$ is second countable* ***stably*** *locally compact space and $\xi_n : \Omega \to X$ is a sequence of random functions. Then for every subsequence of (ξ_n) there exists a further subsequence (ξ'_n) and a random function $\xi : \Omega \to X$ such that $\mathbb{P}_{\xi'_n} \to_w \mathbb{P}_\xi$ in the weak topology of $\mathfrak{M}^1(X^p)$ if and only if for all $\varepsilon > 0$ there exists $Q \in \mathcal{Q}$ such that:*

$$1 - \varepsilon \leq \inf_n \mathbb{P}_{\xi_n}(Q) \leq 1.$$

That is, if $\{\mathbb{P}_{\xi_n}\}_{n \in \mathbb{N}}$ is uniformly tight.

Suppose that (X, d) is a metric space. For every $B \in \mathcal{B}X$ and $\varepsilon > 0$ we define:

$$B_{(\varepsilon)} = \{x \in X \mid d(x, y) < \varepsilon \text{ for some } y \in B\}$$
$$B_{[\varepsilon]} = \{x \in X \mid d(x, y) \leq \varepsilon \text{ for some } y \in B\}$$

The **Lévy-Prohorov metric** ρ_d on $\mathfrak{M}^1(X)$ is defined as (see Chapter 3 in [EK86] for instance):

$$\begin{aligned}\rho_d(\overline{\mu},\overline{\nu}) &:= \inf\{\varepsilon > 0 \mid \overline{\mu}(B) \leq \overline{\nu}(B_{(\varepsilon)}) + \varepsilon \text{ with } B \in \mathcal{B}X\} \\ &= \inf\{\varepsilon > 0 \mid \overline{\mu}(F) \leq \overline{\nu}(F_{(\varepsilon)}) + \varepsilon \text{ with } F \in \mathcal{F}\} \\ &= \inf\{\varepsilon > 0 \mid \overline{\mu}(F) \leq \overline{\nu}(F_{[\varepsilon]}) + \varepsilon \text{ with } F \in \mathcal{F}\}\end{aligned}$$

This definition seems quite asymmetrical; in fact the original definition by Prohorov [Pro56] was $\max(\rho_d(\overline{\mu},\overline{\nu}),\rho_d(\overline{\nu},\overline{\mu}))$ but it can be shown that indeed ρ_d is a metric, thus $\rho_d(\overline{\mu},\overline{\nu}) = \rho_d(\overline{\nu},\overline{\mu})$.

Proposition 7.33 *Suppose that (X,d) is a metric space. Then,*

1. *If X is separable, then $(\mathfrak{M}^1(X),\rho_d)$ is separable.*
2. *Suppose that $(\overline{\mu}_n)$ is a sequence in $\mathfrak{M}^1(X)$ and $\overline{\mu} \in \mathfrak{M}^1(X)$. Then*

   $$\lim_n \rho_d(\overline{\mu}_n,\overline{\mu}) = 0 \textit{ implies that } \overline{\mu}_n \to_w \overline{\mu}.$$

   That is, the topology $\mathcal{T}(\rho_d)$ induced by ρ_d on $\mathfrak{M}^1(X)$ is finer than the weak topology.
3. *If X is Polish, then $(\mathfrak{M}^1(X),\rho_d)$ is Polish and $\mathcal{T}(\rho_d)$ agrees with the weak topology.*

Proof See Theorems 1.7 and 3.1 in Chapter 3 of [EK86]. ■

If (X,d) is a Polish space, then it is also possible to define a metric on $\mathfrak{M}^{\leq 1}(X)$ as

$$\gamma_d(\overline{\mu},\overline{\nu}) = \min(\overline{\mu}(X),\overline{\nu}(X)) \cdot \rho_d(\overline{\mu},\overline{\nu}) + |\overline{\mu}(X) - \overline{\nu}(X)|$$

with similar properties to the Lévy-Prohorov metric (see [DE97, Theorem A.3.3] for instance). The following result by Dudley [Dud68] links the Prohorov metric with the problem of finding probability distributions with given marginals.

Proposition 7.34 *(Dudley) Suppose that (X,d) is a Polish space and $\overline{\mu},\overline{\nu} \in \mathfrak{M}^1(X)$. Then for $\alpha,\beta \geq 0$ the following conditions are equivalent and hold if $\rho_d(\overline{\mu},\overline{\nu}) = \alpha = \beta$.*

1. *$\overline{\mu}(F) \leq \overline{\nu}(F_{[\alpha]}) + \beta$ for all closed $F \in \mathcal{F}$.*
2. *$\overline{\nu}(F) \leq \overline{\mu}(F_{[\alpha]}) + \beta$ for all closed $F \in \mathcal{F}$.*
3. *For every $\varepsilon > 0$ there exists a probability measure $\mathbb{P} \in \mathfrak{M}^1(X \times X)$ such that $\pi_1[\mathbb{P}] = \overline{\mu}$, $\pi_2[\mathbb{P}] = \overline{\nu}$ and $\mathbb{P}(\{(x,y) \mid d(x,y) > \alpha + \varepsilon\}) \leq \beta + \varepsilon$.*

Proof See Proposition 1, Theorem 2 and Corollary 1 in [Dud68]. ■

7.7.2 The Hutchison metric

Let Ω be any set. An **outer measure** on Ω is a σ-subadditive and strict set function $\eta : 2^{\Omega} \to [0, \infty]$; that is,

- $\eta(\emptyset) = 0$.
- $\eta(\bigcup_{n\in\mathbb{N}} A_n) \leq \sum_{n\in\mathbb{N}} \eta(A_n)$ for every sequence (A_n) in 2^{Ω}.

If η is an outer measure on Ω, then by Proposition 1.16 we know that $\mathcal{M}(\eta)$ is a σ-algebra on Ω and η restricted to $\mathcal{M}(\eta)$ is a measure. Now suppose that $(Y, \mathcal{H})$ is a topological space. An outer measure η on Y is said to be **Borel regular** if:

- $\mathcal{B}Y \subseteq \mathcal{M}(\eta)$. That is, every Borel set is η-measurable.
- For any $A \subseteq Y$, there exists $B \in \mathcal{B}Y$ such that $A \subseteq B$ and $\eta(A) = \eta(B)$.

We claim that the second condition is equivalent to $\eta(A) = \inf\{\eta(B) \mid A \subseteq B$ and $B \in \mathcal{B}Y\}$. One direction is obvious. For the non-trivial one suppose that $A \subseteq Y$. Then for every $n \in \mathbb{N}$ there exists $B_n \in \mathcal{B}Y$ such that $A \subseteq B_n$ and $\eta(A) \leq \eta(B_n) < \eta(A) + \frac{1}{n}$. Let $B := \bigcap_n B_n$. Since $\mathcal{B}Y$ is a σ-algebra, then $B \in \mathcal{B}Y$ and clearly $A \subseteq B$ with $\eta(A) = \eta(B)$. This means that for every Borel measure $\overline{\mu} \in \mathfrak{M}(Y)$, the outer measure $\overline{\mu}^* : 2^Y \to [0, \infty]$ is a Borel regular outer measure.

Suppose that (X, d) is a metric space. A subset $B \subseteq X$ is **bounded** if $\sup\{d(x, y) \mid x, y \in B\} < \infty$. Moreover if (X, d) is complete and η is a finite Borel regular outer measure on X, we define the support of η to be the closed set

$$spt(\eta) = X \setminus \bigcup\{G \in \mathcal{G} \mid \eta(G) = 0\}.$$

We set,

$$\begin{aligned}\mathfrak{BR}_+(X) := \{\eta \mid \quad & \eta \text{ Borel regular outer measure on } X,\\ & \eta(X) < \infty \text{ and } spt(\eta) \text{ is bounded}\};\end{aligned}$$

and $\mathfrak{BR}^1(X) = \{\eta \in \mathfrak{BR}_+(X) \mid \eta(X) = 1\}$. Now we can introduce a metric closely related to the Lévy-Prohorov metric. Suppose that (X, d) is a complete metric space. The **Hutchinson metric** [Hut81, Definition 4.3] T_d is defined for $\eta, \theta \in \mathfrak{BR}^1(X)$ as

$$\begin{aligned}T_d(\eta, \theta) = \sup\{\textstyle\int f d\eta - \int f d\theta \mid \quad & f : X \to \mathbb{R} \text{ continuous and}\\ & |f(x) - f(y)| \leq d(x, y) \text{ for all } x, y \in X\};\end{aligned}$$

If (X, d) is a compact metric space, then is X complete (Proposition 1.26-(6)). Furthermore since closed subsets of a compact metric space are compact and hence bounded [Dug66, Chapter IX-4], then for every $\overline{\mu} \in \mathfrak{M}^1(X)$ we have that $spt(\overline{\mu}^*)$ is bounded. Therefore for all $\overline{\mu}, \overline{\nu} \in \mathfrak{M}^1(X)$ we have that $\overline{\mu}^*, \overline{\nu}^* \in \mathfrak{BR}^1(X)$ and

$$T_d(\overline{\mu}^*, \overline{\nu}^*) = \sup\{\textstyle\int f d\overline{\mu} - \int f d\overline{\nu} \mid \begin{array}{l} f : X \to \mathbb{R} \text{ continuous and} \\ |f(x) - f(y)| \leq d(x, y) \text{ for all } x, y \in X\}; \end{array}$$

So if (X, d) is a compact metric space, there is no harm in setting $T_d(\overline{\mu}, \overline{\nu}) := T_d(\overline{\mu}^*, \overline{\nu}^*)$ for all $\overline{\mu}, \overline{\nu} \in \mathfrak{M}^1(X)$; and we have the following result.

Proposition 7.35 *Suppose that (X, d) is a compact metric space. Then,*

1. *$(\mathfrak{M}^1(X), T_d)$ is a compact metric space.*
2. *The topology induced by T_d on $\mathfrak{M}^1(X)$ is the weak topology.*

Proof See Section 9.5 in [Bar88] and Theorem (1) in Section 4.4 of [Hut81]. ■

As it is the case for Polish spaces, if X is a compact metric space, then $\mathfrak{M}^1(X) = \mathfrak{M}^1(X; \tau) = \mathfrak{M}^1(X; R) = \mathfrak{M}^1(X; F)$ (Proposition 1.29-(7)). From Proposition 7.33-(3) and Proposition 7.35-(2) it is immediate then that for compact Polish spaces the Hutchinson and the Lévy-Prohorov metrics are equivalent (i. e., they generate the same topology).

7.7.3 A quasi-metric for valuations

Suppose that $(X, \mathcal{G})$ is a topological space, then it is possible to define a quasimetric (see definition below) $d_{\mathbf{V}}$ on $\mathbf{V}X$, compatible with the order $\sqsubseteq$. Moreover Sünderhauf [Sün98] has shown that $d_{\mathbf{V}}$ can be used to obtain a generalisation of the splitting lemma for simple valuations. We review some of the main results in [Sün98]. Consider a map $d : X \times X \to \mathbb{R}_+$ on a non-empty set X and the following axioms:

$(m1)$ $d(x, x) = 0$

$(m2)$ $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality)

$(m3)$ $d(x, y) = d(y, x)$ (symmetry)

$(m4)$ $d(x, y) = 0 = d(y, x) \Rightarrow x = y$ (separation)

We know from Section 1.12 that if d satisfies $(m1) - (m4)$, then d is called a metric. If d satisfies $(m1)$, $(m2)$, $(m4)$ and we allow d to be ∞, then d

is called a **quasimetric**. More information about the relationship between generalised metrics and topologies can be found in [Kop88]. Let the **truncated substraction** $\dot{-}$ in $\overline{\mathbb{R}}_+$ be given by $x \dot{-} y = \max(x - y, 0)$ for $x, y \in \mathbb{R}_+$ and $\infty \dot{-} y = \infty$ for $y < \infty$ and $x \dot{-} \infty = 0$ for all $x \in \overline{\mathbb{R}}_+$. For $\mu, \nu \in \mathbf{V}X$ let the **sup-distance** on $\mathbf{V}X$ be defined as

$$d_{\mathbf{V}}(\mu, \nu) = \sup\{\mu(G) \dot{-} \nu(G) \mid G \in \mathcal{G}\}.$$

Then it can be shown that $d_{\mathbf{V}}$ is a quasimetric on $\mathbf{V}X$ such that $d_{\mathbf{V}}(\mu, \nu) = 0$ if and only if $\mu \sqsubseteq \nu$. Furthermore, extending the definition from metric spaces for every $r \in \mathbb{R}_+$ we define the open ball centered at μ with radius r as:

$$B(\mu, r) = \{\nu \in \mathbf{V}X \mid d_{\mathbf{V}}(\mu, \nu) < r\}.$$

Let $\mathcal{T}(d_{\mathbf{V}})$ be the topology on $\mathbf{V}X$ generated by the set of balls. Suppose that $(\mu_j)_{j \in J}$ is a net in $\mathbf{V}X$. We say that $(\mu_j)_{j \in J}$ is **forward Cauchy** if for all $\varepsilon > 0$, there exists $i \in J$ such that $d(\mu_j, \mu_k) < \varepsilon$ whenever $i \sqsubseteq j \sqsubseteq k$. A valuation $\mu \in \mathbf{V}X$ is the directed limit of the net $(\mu_j)_{j \in J}$, denoted by $\mu = \lim_j^{\uparrow} \mu_j$, if

$$d(\mu, \nu) = \inf_{j \in J} \sup\{d(\mu_i, \nu) \mid j \sqsubseteq i\} \text{ for all } \nu \in \mathbf{V}X.$$

A subset $H \subseteq \mathbf{V}X$ is Scott open in $(\mathbf{V}X, \sqsubseteq)$ if and only if the following two conditions hold:

- $H \in \mathcal{T}(d_{\mathbf{V}})$.
- If $(\mu_j)_{j \in J}$ is a forward Cauchy net and $\lim_j^{\uparrow} \mu_j \in H$, then there exists $j \in J$ such that $B(\mu_i, r) \subseteq H$ for all $j \sqsubseteq i$.

Let $\sqsubseteq_s$ denote the specialisation preorder in $(X, \mathcal{G})$ and define the weight function w for all $x, y \in X$, $r, s \in \overline{\mathbb{R}}_+$ and η simple valuation in $\mathbf{V}X$ as:

$$w(r\delta_x + \eta, r\delta_y + \eta) := \begin{cases} 0 & \text{if } x \sqsubseteq_s y \\ r & \text{otherwise} \end{cases}$$

$$w(r\delta_x + \eta, s\delta_x + \eta) := r \dot{-} s.$$

Now suppose that μ, ν are simple valuations in $\mathbf{V}X$. Let $[\mu, \nu]$ denote the set of all finite tuples $(\eta_0, \eta_1, \ldots, \eta_k)$ of simple valuations in $\mathbf{V}X$ such that $\mu = \eta_0$, $\nu = \eta_k$, and (η_j, η_{j+1}) is in the domain of w for all $j = 0, \ldots, k-1$. Set

$$d_S(\mu, \nu) := \inf\{\textstyle\sum_{j=0}^{k-1} w(\eta_j, \eta_{j+1}) \mid (\eta_0, \eta_1, \ldots, \eta_k) \in [\mu, \nu] \text{ and } k \in \mathbb{N}\}$$

Proposition 7.36 *(Sünderhauf's Generalised Splitting Lemma) Suppose that* $(X, \mathcal{G})$ *is a topological space and* μ, ν *are simple valuations in* $\mathbf{V}X$. *Then,*

1. $d_{\mathbf{V}}(\mu, \nu) = d_S(\mu, \nu)$.

2. $\mu \sqsubseteq \nu$ *if and only if there is a finite chain of simple valuations* $\mu = \eta_0 \sqsubseteq \eta_1 \sqsubseteq \ldots \sqsubseteq \eta_k = \nu$ *such that for every* $j = 0, \ldots, k-1$,

   - $\eta_j = r\delta_x + \eta \sqsubseteq r\delta_y + \eta = \eta_{j+1}$ *with* $x \sqsubseteq_s y$ *for some* $x, y \in X$ *and* η *simple valuation; or*

   - $\eta_j = r\delta_x + \eta \sqsubseteq s\delta_x + \eta = \eta_{j+1}$ *with* $r \leq s$ *for some* $r, s \in \overline{\mathbb{R}}_+$ *and* η *simple valuation.*

Proof See Theorem 2.4 and Corollary 2.6 in [Sün98]. ■

As a corollary one obtains the splitting lemma for simple valuations (Proposition 7.12), hence Sünderhauf decided to name his result "Generalised Splitting Lemma". This generalisation seems to be unrelated to the generalisation of the splitting lemma presented in Section 7.5. Nevertheless, since the patch space of a second countable locally compact sober space $(X, \mathcal{G})$ is a Polish space (Proposition 3.10-(5)), then Proposition 7.34 and Proposition 7.15 suggest that there should exist a stronger link between both results (at least for ω-continuous dcpo's).

Similarly from Propositions 7.28 and 7.32 it seems reasonable that for a metric d on $(X, \mathcal{G}_p)$ inducing the patch topology $\mathcal{G}_p$, it should be possible to define quasi-metrics on $\mathbf{P}^1 X$ and $\mathbf{P}^1 X^{cc}$ (related under some form of duality) which recover the Lévy-Prohorov metric ρ_d on $\mathfrak{M}^1(X^p)$. We will not undertake such tasks here but rather leave them as possible directions for future work. Similar ideas have been explored by A. J. Baddeley [Bad91] for capacities.

Problem 7.37 *Suppose that* $(X, \mathcal{G})$ *is a Polish space and* $(D, \sqsubseteq)$ *is an* ω-*continuous MP-hull for* X *(see Section 5.3.4). Then* $(D, \mathcal{G}_\lambda D)$ *is a Polish space. Given a metric* d *on* X *inducing the topology* $\mathcal{G}$, *is it possible to extend it to a metric* $\mathbf{d}$ *on* D *which induces the Lawson topology* $\mathcal{G}_\lambda D$ *? (note that the converse is always possible) Is the Lévy-Prohorov metric* ρ_d *on* $\mathfrak{M}^1 X$ *the restriction of a quasi-metric on* $\mathbf{P}^1 X$ *? Is* $\rho_{\mathbf{d}}$ *an extension of* ρ_d *?*

8. Interval-valued functions: integration and maximality

8.1 Summary

Edalat and Escardó [EE96] have constructed an interval Riemann integral using domain theory with applications to the programming language PCF. This integral was later extended by the author [AM96] to an interval Riemann-Stieltjes integral and an interval version of the R-integral [Eda95b] for compact metric spaces. In the light of Theorem 6.40 and Howroyd's extension of the Riemann-Edalat integral we can construct interval Lebesgue, valuation and Riemann-Edalat integrals (Theorem 8.11) which generalise the constructions in Chapter 3 of [AM96].

We start by reviewing some properties of complete continuous semilattices and exhibit a broad class of functions that are always maximal continuous (Lemma 8.7). The second part of the chapter gives a characterisation of the maximal functions in $[\mathbf{I}\overline{\mathbb{R}} \to \mathbf{I}\overline{\mathbb{R}}]$, where $\mathbf{I}\overline{\mathbb{R}}$ is the domain of compact extended-real intervals (Theorems 8.17 and 8.25 and Corollaries 8.28 and 8.29). We finish with a discussion on which functions $f : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ admit a maximal continuous extension to $[\mathbf{I}\overline{\mathbb{R}} \to \mathbf{I}\overline{\mathbb{R}}]$ and the relation of this property with integration.

8.2 Auxiliary results on complete continuous semilattices

A poset $(L, \sqsubseteq)$ is called a **complete continuous semilattice**, if $(L, \sqsubseteq)$ is a continuous semilattice and $\bigwedge E$ exists for all nonempty $E \subseteq L$ [GHK$^+$80, Definitions 0-2.11, I-1.26].

Proposition 8.1 *Suppose that $(L, \sqsubseteq)$ is a complete continuous semilattice. If $(x_j)_{j\in J}$ is a net in L and $x \in L$, then*

$$(x_j) \to_\sigma x \text{ if and only if } x \sqsubseteq \liminf_j x_j \equiv \textstyle\bigsqcup^{\uparrow}_{j\in J} \bigwedge\{x_k \mid j \sqsubseteq k\}.$$

Proof See Theorem II-1.8 and Exercise II-1.18 in [GHK$^+$80]. ■

Corollary 8.2 *Suppose that $(X, \mathcal{G})$ is any topological space and $(L, \sqsubseteq)$ is a complete continuous semilattice. If $f : X \to L$ and $x \in X$, then the following conditions are equivalent:*

1. f is continuous at x.

2. For every net $(x_j)_{j\in J}$ in X with $(x_j) \to x$, $f(x) \sqsubseteq \liminf_j f(x_j)$.

Proof This follows immediately from part (4) of Proposition 1.9 and Proposition 8.1. ■

The following examples for the domain of intervals $\mathbf{I}\mathbb{R}$ (defined in the next section) give us more insight to the previous result.

Example 8.3 *Let $f, g : \mathbb{R}^\lambda \to \mathbf{I}\mathbb{R}$ be given by*

$$f(x) = \begin{cases} \{\frac{1}{2}\} & \textit{if } x < 0 \\ [0,1] & \textit{otherwise} \end{cases}$$

and

$$g(x) = \begin{cases} \{\frac{1}{2}\} & \textit{if } x \leq 0 \\ [0,1] & \textit{otherwise} \end{cases}$$

*Then f is continuous but g is **not** continuous at 0.*

Example 8.4 *Let $f : \mathbb{R}^\lambda \to \mathbf{I}\mathbb{R}$ be given by*

$$f(x) = \begin{cases} [-1,1] & \textit{if } x = 0 \\ [-\frac{1}{2}, \frac{1}{2}] & \textit{otherwise} \end{cases}$$

Then f is continuous.

The following result extends the result for injective spaces given in Section 6.2.2.

Proposition 8.5 *Suppose that $(X, \mathcal{G})$ and $(Y, \mathcal{H})$ are T_0 topological spaces, $e : X \to Y$ is a **dense** subspace embedding and $(L, \sqsubseteq)$ is a complete continuous semilattice. Then, any continuous function $f \in [X \to L^\sigma]$ has a greatest continuous extension $f_e : Y \to L$ of f along e given by the formula*

$$f_e(y) := \bigsqcup\nolimits^{\uparrow}\{\bigwedge f[e^{-1}G] \mid G \in \mathcal{N}(y)\} \textit{ for all } y \in Y.$$

Proof See Exercise II-3.19 in [GHK+80]. ■

Edalat and Escardó [EE96, Proposition 3.3] have shown a specialized version of this result for metric spaces.

In the following $(Y, \mathcal{H})$ is a T_0 topological space and $(L, \sqsubseteq)$ is a complete continuous semilattice. For $f : Y \to L$ define the sets of continuous points and continuous maximal points of f as,

$$\begin{aligned} C_f &:= \{x \mid f \text{ is continuous at } x\} \\ S_f &:= C_f \cap f^{-1}\max(L) \end{aligned}$$

Definition 8.6 *Let $f : Y \to L$ be any function such that S_f is a dense subset of Y. We define the* ***maximal continuous representation*** *of f as the function $f_c : Y \to L$ given by*

$$f_c(y) := \bigsqcup^{\uparrow}\{\bigwedge f[G \cap S_f] \mid G \in \mathcal{N}(y)\}.$$

Notice that since S_f is dense on Y then $G \cap S_f \neq \emptyset$ for all $G \in \mathcal{N}(y)$. Hence f_c is well defined.

Suppose that $(J, \sqsubseteq)$ is a preordered set where $i \sqsubseteq j$ for all $i, j \in J$; that is, the trivial preorder. Then, a net $(x_j)_{j\in J}$ in Y is called a **diffuse net**. The next result shows that if $f : Y \to L$ is any function such that S_f is dense in Y, then f_c is always continuous and maximal.

Lemma 8.7 *Suppose that $(Y, \mathcal{H})$ is a T_0 topological space and $(L, \sqsubseteq)$ is a complete continuous semilattice. Let $f : Y \to L$ be any function such that S_f is dense in Y. Then*

1. *f_c is continuous and agrees with f in S_f. If $h \in [Y \to L^\sigma]$ and $h(x) = f(x)$ for all $x \in S_f$, then $h \sqsubseteq f_c$.*
2. *The set S_{f_c} is dense in Y and $S_f \subseteq S_{f_c}$.*
3. *$f_c \in \max[Y \to L^\sigma]$ and $(f_c)_c = f_c$.*
4. *If f is continuous, then $f \sqsubseteq f_c$. In fact, f_c is the only maximal continuous function above f.*
5. *For all $y \in Y$, there exists a net $(x_j)_{j\in J}$ in S_f such that $(x_j) \to y$ and $f_c(y) = \liminf_j f(x_j)$.*
6. *$f_c(y) = \bigwedge\{\liminf_j f(x_j) \mid (x_j)_{j\in J}$ net in S_f with $(x_j) \to y\}$ for every $y \in Y$.*

Proof For simplicity for every $y \in Y$ define

$$Net(y) := \{(x_j)_{j\in J} \text{ net in } S_f \mid (x_j) \to y\}.$$

1. Let $X := S_f$ and $\mathcal{G}$ be the subspace topology induced from Y. Then $f\restriction_X : X \to L$ is a continuous function and X is a dense subspace of Y. The claim follows from Proposition 8.5 defining $e : X \to Y$ as $e(x) = x$ for all $x \in X$ and $f_c := (f\restriction_X)_e$.
2. By part (1) we know that f_c agrees with f in S_f, hence $f_c(x) \in \max(L)$

for all $x \in S_f$. But f_c is continuous in Y hence $S_f \subseteq S_{f_c}$. Since S_f is dense in Y the claim follows.

3. Suppose that $h \in [Y \to L]$ and $f_c \sqsubseteq h$. If $x \in S_f$, then by part (1) we have that $f_c(x) = f(x) \in \max(L)$. Hence $h(x) = f(x)$ for all $x \in S_f$. Again by part (1) $h \sqsubseteq f_c$, hence $f_c = h$. It follows that f_c is maximal. By part (2) we know that $S_f \subseteq S_{f_c} \subseteq S_{(f_c)_c}$ and $f(x) = f_c(x) = (f_c)_c(x)$ for all $x \in S_f$. Then by part (1), $(f_c)_c \sqsubseteq f_c$. The conclusion follows since $(f_c)_c$ is also maximal.

4. $f \sqsubseteq f_c$ is immediate from parts (1) and (3). If $h \in \max[Y \to L]$ and $f \sqsubseteq h$, then $h(x) = f(x)$ for all $x \in S_f$. Again from part (1) we have that $h \sqsubseteq f_c$. Since h is maximal then $h = f_c$.

5. Fix $y \in Y$. For every $G \in \mathcal{N}(y)$ let $\mathbf{k}_G := \bigwedge f[G \cap S_f]$ and $(x_{(G,j)})_{j \in J_G}$ be a diffuse net covering all the elements of $G \cap S_f$; that is, $G \cap S_f = \{x_{(G,j)} \mid j \in J_G\}$. Note that the latter construction is always possible; in fact, we could choose J_G to be $G \cap S_f$ itself. Obviously $\mathbf{k}_G = \bigwedge_{j \in J} f(x_{(G,j)})$. Now let $J := \{(G,j) \mid G \in \mathcal{N}(y) \text{ and } j \in J_G\}$ and define the following preorder $\leq$ in J :

$$(G,j) \leq (H,i) \text{ if } G \supseteq H.$$

Clearly $(x_{(G,j)})_{(G,j) \in J}$ defines a net. It follows that $x_{(G,j)} \to y$ since for every $G \in \mathcal{N}(y)$ we have that $x_{(H,i)} \in G$ for all $(H,i) \geq (G,j)$, where j is an arbitrary element of J_G. Moreover, since $\mathbf{k}_G = \bigwedge_j f(x_{(G,j)})$ and the preorder in J_G is trivial, the latter argument also shows that $\mathbf{k}_G = \bigwedge\{f(x_{(H,i)}) \mid (H,i) \geq (G,j)\}$ for any $j \in J_G$. Therefore

$$\begin{aligned} f_c(y) &\equiv \textstyle\bigsqcup^{\uparrow}_{G \in \mathcal{N}(y)} \mathbf{k}_G \\ &= \textstyle\bigsqcup^{\uparrow}_{G \in \mathcal{N}(y)} \bigwedge\{f(x_{(H,i)}) \mid (H,i) \geq (G,j)\} \\ &\qquad \text{where } j \in J_G \text{ is arbitrary} \\ &= \textstyle\bigsqcup^{\uparrow}_{m \in J} \bigwedge\{f(x_k) \mid k \geq m\} \\ &\equiv \liminf_m f(x_m). \end{aligned}$$

6. $\boxed{\sqsubseteq}$ Fix $y \in Y$. Suppose that $(x_j)_{j \in J} \in Net(y)$. Since $x_j \in S_f$, by part (1) we have that $f_c(x_j) = f(x_j)$ for all $j \in J$. Since $(x_j) \to y$ and f_c is continuous, from Corollary 8.2 it follows that $f_c(y) \sqsubseteq \liminf_j f_c(x_j) = \liminf_j f(x_j)$. Therefore $f_c(y) \sqsubseteq \bigwedge\{\liminf_j f(x_j) \mid (x_j)_{j \in J} \in Net(y)\}$. $\boxed{\sqsupseteq}$ In part (5) we constructed a net $(x_j)_{j \in J} \in Net(y)$ with $f_c(y) = \liminf_j f(x_j)$. ■

Let $Con : \{f : Y \to L \mid cl(S_f) = Y\} \to [Y \to L]$ be defined as $Con(f) := f_c$.

8.3 A review of the domain of intervals

We make a brief review of the domains of intervals and their main properties; the reader is referred to Section 3 of [EE96] and [AM96] for more information. We let $\mathbf{I}\mathbb{R}$ be the set of compact intervals in $\mathbb{R}$ with added least element $\perp = [-\infty, \infty]$. Similarly $\mathbf{I}\overline{\mathbb{R}}$ denotes the set of compact intervals in $\overline{\mathbb{R}}$; notice that the intervals $[a, \infty]$ and $[-\infty, b]$ are elements of $\mathbf{I}\overline{\mathbb{R}}$. For $\mathbf{x}, \mathbf{y} \in \mathbf{I}\overline{\mathbb{R}}$ we set $\mathbf{x} \sqsubseteq \mathbf{y}$ if $\mathbf{x} \supseteq \mathbf{y}$. Then $(\mathbf{I}\overline{\mathbb{R}}, \sqsubseteq)$ is a pointed ω-continuous dcpo where $\mathbf{x} \ll \mathbf{y}$ if and only if $int(\mathbf{x}) \supseteq \mathbf{y}$. Notice that $(\mathbf{I}\overline{\mathbb{R}}, \sqsubseteq)$ is a subdcpo of the upper space $(\mathbf{U}\overline{\mathbb{R}}, \sqsubseteq)$ which is also an MP-hull for $\overline{\mathbb{R}}$ (see Section 5.3.4). For $\mathbf{x} \in \mathbf{I}\overline{\mathbb{R}}$ we let $\underline{\mathbf{x}} := \inf \mathbf{x}$ and $\overline{\mathbf{x}} := \sup \mathbf{x}$; thus, $\mathbf{x} = [\underline{\mathbf{x}}, \overline{\mathbf{x}}]$. A countable basis for $\mathbf{I}\overline{\mathbb{R}}$ is given by the set $\{\uparrow\mathbf{x} \mid \underline{\mathbf{x}}, \overline{\mathbf{x}} \in \mathbb{Q} \cup \{-\infty, \infty\}\}$. For $[a, b] \in \mathbf{I}\overline{\mathbb{R}}$, we let $\Box(a, b) = \{\mathbf{x} \in \mathbf{I}\mathbb{R} \mid \mathbf{x} \subseteq (a, b)\}$; this definition extends to $\Box[-\infty, b)$ and $\Box(a, \infty]$ with the obvious meaning. Hence the set $\{\Box(p, q) \mid p, q \in \mathbb{Q} \text{ with } p < q\}$ is a countable basis for $\mathbf{I}\mathbb{R}$. For any $[a, b] \in \mathbf{I}\overline{\mathbb{R}}$, we let $\mathbf{I}[a, b] = \{\mathbf{x} \in \mathbf{I}\mathbb{R} \mid \mathbf{x} \subseteq [a, b]\}$. Hence $(\mathbf{I}[a, b], \sqsubseteq)$ is a subdcpo of $(\mathbf{I}\overline{\mathbb{R}}, \sqsubseteq)$ with $[a, b]$ as the least element. Suppose that $\mathbf{x} \in \mathbf{I}\overline{\mathbb{R}}$, then

$$|\mathbf{x}| := \begin{cases} \overline{\mathbf{x}} - \underline{\mathbf{x}} & \text{if } \overline{\mathbf{x}}, \underline{\mathbf{x}} \in \mathbb{R} \\ \infty & \text{otherwise.} \end{cases}$$

Remark 8.8 *Notice that* $|\{\infty\}| = |\{-\infty\}| = \infty$. *Hence* $|\_| : \mathbf{I}\overline{\mathbb{R}} \to [0, \infty]^{\geq}$ *is a continuous function. In particular, the sequence* $(\{n\})_{n \in \mathbb{N}}$ *converges in the Scott topology of* $\mathbf{I}\overline{\mathbb{R}}$ *to* $\{\infty\}$ *and* $\infty = |\{\infty\}| \sqsubseteq 0 = \liminf_n |\{n\}|$, *where* $\sqsubseteq$ *is* $\geq$.

Define the following operations for all $\mathbf{x}, \mathbf{z} \in \mathbf{I}\overline{\mathbb{R}}$ and $r \in \mathbb{R}_+$,

- $r \cdot \mathbf{x} = [r\underline{\mathbf{x}}, r\overline{\mathbf{x}}]$.
- $\mathbf{x} + \mathbf{z} = [\underline{\mathbf{x}} + \underline{\mathbf{z}}, \overline{\mathbf{x}} + \overline{\mathbf{z}}]$
- $\infty + \infty = \infty$; $r \cdot \infty = \infty$; $r \cdot (-\infty) = -\infty$; $-\infty + (-\infty) = -\infty$
- $\infty + (-\infty) = \perp$; $r \cdot \perp = \perp$ and $\perp + \perp = \perp$

Then $(\mathbf{I}\overline{\mathbb{R}}, +, \{0\}, \cdot)$ is a real cone where the operations $+ : \mathbf{I}\overline{\mathbb{R}} \times \mathbf{I}\overline{\mathbb{R}} \to \mathbf{I}\overline{\mathbb{R}}$ and $\cdot : \mathbb{R}_+^{\lambda} \times \mathbf{I}\overline{\mathbb{R}} \to \mathbf{I}\overline{\mathbb{R}}$ are continuous. Nevertheless $(\mathbf{I}\overline{\mathbb{R}}, +, \{0\}, \cdot)$ is **not** a topological cone (see Section 6.2.5) since $\mathbb{R}_+$ is topologised with the metric topology not with the Scott topology.

It is not difficult to see that $(\mathbf{I}\overline{\mathbb{R}}, \sqsubseteq)$ is a complete continuous semilattice where $\bigwedge E = [\inf\{\underline{\mathbf{x}} \mid \mathbf{x} \in E\}, \sup\{\overline{\mathbf{x}} \mid \mathbf{x} \in E\}]$ for every nonempty $E \subseteq \mathbf{I}\overline{\mathbb{R}}$.

Suppose that $(X, \mathcal{G})$ is a topological space and $\mathbf{f} : X \to \mathbf{I}\overline{\mathbb{R}}$. We define $\underline{\mathbf{f}}$, $\overline{\mathbf{f}} : X \to \overline{\mathbb{R}}$ as

$$\begin{aligned}\overline{\mathbf{f}}(x) &= \overline{\mathbf{f}(x)} = \sup \mathbf{f}(x)\\ \underline{\mathbf{f}}(x) &= \underline{\mathbf{f}(x)} = \inf \mathbf{f}(x).\end{aligned}$$

Then $\mathbf{f} : X \to \mathbf{I}\overline{\mathbb{R}}$ is Scott continuous if and only if $\overline{\mathbf{f}}$ is USC and $\underline{\mathbf{f}}$ is LSC. In other words

$$\mathbf{f} \in [X \to \mathbf{I}\overline{\mathbb{R}}] \Leftrightarrow \begin{cases} \overline{\mathbf{f}} \in [X \to [-\infty, \infty]^{\geq}] \\ \underline{\mathbf{f}} \in [X \to [-\infty, \infty]^{\leq}]. \end{cases}$$

Conversely, if $f_1 \in [X \to [-\infty, \infty]^{\leq}]$, $f_2 \in [X \to [-\infty, \infty]^{\geq}]$ with $f_1 \leq f_2$, then the function $[f_1, f_2] : X \to \mathbf{I}\overline{\mathbb{R}}$ given by $x \mapsto [f_1(x), f_2(x)]$ for all $x \in X$, is Scott continuous [EE96, Lemma 3.1]. Recall the definition of the upper and lower semicontinuous envelopes from Section 6.2.1. It follows that if $f : X \to \overline{\mathbb{R}}$, then $f^{\square}(x) := [f_E, f^E]$ is Scott continuous. Thus, for a function $\mathbf{f} : X \to \mathbf{I}\overline{\mathbb{R}}$ we define the greatest interval Scott continuous envelope of $\mathbf{f}$ as $\mathbf{f}_E \in [X \to \mathbf{I}\overline{\mathbb{R}}]$ given by

$$\mathbf{f}_E := [\underline{\mathbf{f}}_E, \overline{\mathbf{f}}^E].$$

Clearly $\mathbf{f}_E \sqsubseteq \mathbf{f}$, and $\mathbf{f}_E$ is the greatest Scott continuous function below $\mathbf{f}$. Thus, $\mathbf{f}$ is Scott continuous if and only if $\mathbf{f}_E = \mathbf{f}$. Now observe that from Propositions 8.1 and 8.5 we can derive the following corollary.

Corollary 8.9 *Suppose that $(X, \mathcal{G})$ and $(Y, \mathcal{H})$ are T_0 topological spaces, $e : X \to Y$ is a **dense** subspace embedding. Then,*

1. *if $(\mathbf{x}_j)_{j \in J}$ is a net in $\mathbf{I}\overline{\mathbb{R}}$. Then* lim inf *of $(\mathbf{x}_j)$ exists in $(\mathbf{I}\overline{\mathbb{R}}, \sqsubseteq)$ and is given by*

$$\underline{\lim}_j \mathbf{x}_j := [\liminf_j \underline{\mathbf{x}_j}, \limsup_j \overline{\mathbf{x}_j}]$$

where $\liminf_j \underline{\mathbf{x}_j}$ and $\limsup_j \overline{\mathbf{x}_j}$ are taken in the continuous lattice $(\overline{\mathbb{R}}, \leq)$.

2. *A continuous function $\mathbf{f} : X \to \mathbf{I}\overline{\mathbb{R}}$ has a greatest Scott continuous extension $\mathbf{f}_e \in [Y \to \mathbf{I}\overline{\mathbb{R}}]$ to Y along e given by*

$$\mathbf{f}_e(y) = [\sup\{\inf \underline{\mathbf{f}}[e^{-1}G] \mid G \in \mathcal{N}(y)\}, \inf\{\sup \overline{\mathbf{f}}[e^{-1}G] \mid G \in \mathcal{N}(y)\}]$$

Proof

1. Let

$$\mathbf{z}_k := \bigwedge\{\mathbf{x}_k \mid j \sqsubseteq k\} = [\inf\{\underline{\mathbf{x}_k} \mid j \sqsubseteq k\}, \sup\{\overline{\mathbf{x}_k} \mid j \sqsubseteq k\}].$$

Hence $j \sqsubseteq m$ implies that $\mathbf{z}_j \supseteq \mathbf{z}_m$. Hence the set $\{\mathbf{z}_j \mid j \in J\}$ is directed in $(\mathbf{I}\overline{\mathbb{R}}, \sqsubseteq)$ and $\bigsqcup^{\uparrow}_j \mathbf{z}_j$ is given by:

$$\begin{aligned} \textstyle\bigsqcup^{\uparrow}_j \mathbf{z}_j &= [\sup_j \inf\{\underline{\mathbf{x}_k} \mid j \sqsubseteq k\}, \inf_j \sup\{\overline{\mathbf{x}_k} \mid j \sqsubseteq k\}] \\ &\equiv [\liminf_j \underline{\mathbf{x}_j}, \limsup_j \overline{\mathbf{x}_j}]. \end{aligned}$$

2. Immediate from Proposition 8.5.

■

Suppose that $\mathbf{f} : X \to \mathbf{I}\overline{\mathbb{R}}$ is any function. Then if $(Y, \mathcal{H})$ is a topological space and $e : X \to Y$ is a dense subspace embedding, we define the greatest Scott continuous function $\mathbf{f}_Y$ below $\mathbf{f}$ along e as

$$\begin{aligned} \mathbf{f}_Y(y) &:= [\underline{\mathbf{f}}_Y(y), \overline{\mathbf{f}}^Y(y)] \\ &= [\sup\{\inf \underline{\mathbf{f}}_E[e^{-1}G] \mid G \in \mathcal{N}(y)\}, \inf\{\sup \overline{\mathbf{f}}^E[e^{-1}G] \mid G \in \mathcal{N}(y)\}] \\ &= \textstyle\bigsqcup^{\uparrow}\{\bigwedge \mathbf{f}_E[e^{-1}G] \mid G \in \mathcal{N}(y)\} \end{aligned}$$

In particular, if $(D, \sqsubseteq, e)$ is a dense S-domain representation for X we have that

$$\begin{aligned} \mathbf{f}_D &= [\underline{\mathbf{f}}_D, \overline{\mathbf{f}}^D] \\ &= \textstyle\bigsqcup^{\uparrow}\{\bigwedge \mathbf{f}_E[e^{-1}{\uparrow}z] \mid z \ll y \text{ and } z \in D\}. \end{aligned}$$

8.4 Interval integrals

Suppose that $\mathbf{f} : X \to \mathbf{I}\mathbb{R}$ is such that $\underline{\mathbf{f}}$ and $\overline{\mathbf{f}}$ are Borel measurable, and $\mu \in \mathfrak{M}^1(X)$. Then $\underline{\mathbf{f}} \leq \overline{\mathbf{f}}$ and we can define the **Lebesgue interval integral** of F with respect to μ as

$$\textstyle\int \mathbf{f} d\mu = [\int \underline{\mathbf{f}} d\mu, \int \overline{\mathbf{f}} d\mu].$$

We say that $\mathbf{f} : X \to \mathbf{I}\overline{\mathbb{R}}$ is bounded if there exists $[-M_{\mathbf{f}}, M_{\mathbf{f}}] \subseteq \mathbb{R}$ such that $[-M_{\mathbf{f}}, M_{\mathbf{f}}] \sqsubseteq \mathbf{f}(x)$ for all $x \in X$. Let $\lceil X \to \mathbf{I}\mathbb{R}\rceil$ be the subdcpo of bounded functions in $[X \to \mathbf{I}\mathbb{R}]$. If $\nu \in \mathbf{P}^1 X$ and $\mathbf{f} \in \lceil X \to \mathbf{I}\mathbb{R}\rceil$, then we define the **interval valuation integral** of $\mathbf{f}$ with respect to ν as

$$V(\mathbf{f}, \nu) = [V^l(\underline{\mathbf{f}}, \nu), V^u(\overline{\mathbf{f}}, \nu)].$$

From the results in Chapter 6 it is clear that if $f : X \to \mathbb{R}$ is bounded and ν extends to a Borel probability measure $\overline{\nu}$, then $V(f^{\square}, \nu) = \int f^{\square} d\overline{\nu}$. Hence if f is V-integrable, then $V(f^{\square}, \nu) = \{\int f d\overline{\nu}\}$. For a bounded $\mathbf{f} : X \to \mathbf{I}\mathbb{R}$ we say that $\mathbf{f}$ is V-integrable if $V(\mathbf{f}_E, \nu)$ is a singleton. Combining Theorems 6.37 and 6.38 we obtain immediately the following properties:

1. $V : [X \to \mathbf{I}[-M, M]]^{is} \times \mathbf{P}^1 X \to \mathbf{I}\mathbb{R}$ is continuous in each argument

for every $M \in \mathbb{R}$.

2. Fix $\nu \in \mathbf{P}^1 X$. Suppose that $\langle \mathbf{f}_j \rangle_{j \in J}$ is a monotone net in $\lceil X \to \mathbf{I}\mathbb{R} \rceil$. Then $\bigsqcup_j^{\uparrow} \mathbf{f}_j \in \lceil X \to \mathbf{I}\mathbb{R} \rceil$ and $V(\bigsqcup_j^{\uparrow} \mathbf{f}_j, \nu) = \bigsqcup_j^{\uparrow} V(\mathbf{f}_j, \nu)$.

3. Fix $\mathbf{f} \in \lceil X \to \mathbf{I}\mathbb{R} \rceil$. Then $V(\mathbf{f}, \_) : \mathbf{P}^1 X \to \mathbf{I}\mathbb{R}$ is (Scott) continuous.

4. $V : \lceil X \to \mathbf{I}\mathbb{R} \rceil \times \mathbf{P}^1 X \to \mathbf{I}\mathbb{R}$ is linear in the first argument and linear under convex combinations in the second argument (note that the scalars belong to $\mathbb{R}_+$).

5. Suppose that $(Y, \mathcal{H})$ is a topological space, $\nu \in \mathbf{P}^1 X$, $h \in [X \to Y]$ and $\mathbf{f} \in \lceil X \to \mathbf{I}\mathbb{R} \rceil$. Then $V_X(\mathbf{f} \circ h, \nu) = V_Y(\mathbf{f}, \nu_h)$ where $\nu_h \equiv \nu \circ h^{-1}$.

6. The topology $\mathcal{G}_{pw}(\mathbf{P}^1 X)$ in $\mathbf{P}^1 X$ is the weakest topology such that the function $V(\mathbf{f}, \_) : \mathbf{P}^1 X \to \mathbb{R}^{\leq}$ is Scott continuous for all $\mathbf{f} \in \lceil X \to \mathbf{I}\mathbb{R} \rceil$.

Now suppose that $(X, \mathcal{G})$ is a Hausdorff topological space, $(D, \sqsubseteq, e, \perp)$ a pointed S-domain representation for X and $\mu \in \mathfrak{M}^1(X; \tau)$. If $\mathbf{f} : X \to \mathbf{I}\mathbb{R}$ is bounded, then we can define the **interval R-integral** as:

$$R(\mathbf{f}, \mu) = [R^l(\underline{\mathbf{f}}, \mu), R^u(\overline{\mathbf{f}}, \mu)].$$

The following properties are easy consequences of Proposition 6.27. Suppose that $\mathbf{f}, \mathbf{g} : X \to \mathbf{I}\mathbb{R}$, $\mu, \mu_1, \mu_2 \in \mathfrak{M}^1(X; \tau)$ and $r, s \in \mathbb{R}_+$, then

1. $R(r\mathbf{f} + s\mathbf{g}, \mu) \sqsupseteq rR(\mathbf{f}, \mu) + sR(\mathbf{g}, \mu)$.

2. If $r \in [0, 1]$, then $R(\mathbf{f}, r\mu_1 + (1-r)\mu_2) = rR(\mathbf{f}, \mu_1) + (1-r)R(\mathbf{f}, \mu_2)$.

3. If $\nu = \sum_{x \in |\nu|} r_x \delta_x$ is any simple valuation in $\mathbf{P}^1 D$, then for $\mathbf{f} : D \to \mathbf{I}\mathbb{R}$ we have that $\int \mathbf{f} d\overline{\nu} = [\sum_{x \in |\nu|} r_x \underline{\mathbf{f}}(x), \sum_{x \in |\nu|} r_x \overline{\mathbf{f}}(x)]$.

4. If $\nu_1, \nu_2 \in S(\mu_e)$ and $\nu_1 \sqsubseteq \nu_2$, then $\int \mathbf{f}_D d\overline{\nu}_1 \sqsubseteq \int \mathbf{f}_D d\overline{\nu}_2$ (see remark below).

5. $R(\mathbf{f}, \mu) = \bigsqcup_{\nu \in S(\mu_e)}^{\uparrow} \int \mathbf{f}_D d\overline{\nu}$.

6. f is R-integrable if and only if $\mu(D_f) = 0$. In which case $R(f^{\square}, \mu) = \{\int f d\mu\}$.

Remark 8.10 *If X is not dense in D, then (see Remark 6.6) it does not follow that $\underline{\mathbf{f}}_D(y) \leq \overline{\mathbf{f}}^D(y)$ for all $y \in D$. Therefore $\mathbf{f}_D(y)$ is not well defined unless $e^{-1}G \neq \emptyset$ for all $G \in \mathcal{N}(y)$. This does not affect the construction of the R-integral and the reason is as follows. Suppose that $\nu := \sum_{y \in |\nu|} r_y \delta_y \in S(\mu_e)$. Then $\nu \ll \mu_e$. Choose any $y \in |\nu|$. If $G \in \mathcal{N}(y)$, then $0 < r_y \leq \nu(G) \leq \mu_e(G)$. Since μ is a measure this implies that $e^{-1}G \neq \emptyset$. Thus,*

$\mathbf{f}_D(y)$ *is well defined for all* $y \in |\nu|$ *and* $\nu \ll \mu_e$.

As an immediate corollary to Theorem 6.40 we have

Theorem 8.11 *Let* $(X, \mathcal{G})$ *be a Hausdorff topological space. Let* $(D, \sqsubseteq)$ *be a pointed continuous dcpo and* $e : X \to D^\sigma$ *a dense subspace embedding. Let* μ *be a* τ*-smooth probability measure on* $\mathcal{B}X$. *Then for any bounded function* $\mathbf{f} : X \to \mathbf{I}\mathbb{R}$ *we have that*

$$R(\mathbf{f}, \mu) = \textstyle\int_X \mathbf{f}_E d\mu = V(\mathbf{f}_E, \mu_\uparrow) = V(\mathbf{f}_D, \mu_e) = \int_D \mathbf{f}_D d\overline{\mu}_e$$

$R(\mathbf{f}, \mu)$ *is a singleton if and only if* $\int_X \underline{\mathbf{f}} d\mu = \int_X \overline{\mathbf{f}} d\mu$. *That is,* $\mu(\{x \mid \underline{\mathbf{f}}(x) \neq \overline{\mathbf{f}}(x)\}) = 0$.

Proof Notice that by definition $\mathbf{f}_E \equiv [(\underline{\mathbf{f}})_E, (\overline{\mathbf{f}})^E]$, hence $\underline{\mathbf{f}_E} = (\underline{\mathbf{f}})_E$ and $\overline{\mathbf{f}_E} = (\overline{\mathbf{f}})^E$. Similarly $\underline{\mathbf{f}_D} = (\underline{\mathbf{f}})_D$ and $\overline{\mathbf{f}_D} = (\overline{\mathbf{f}})^D$. Hence by Theorem 6.40 we have that

$$\begin{array}{lll} R^l_X(\underline{\mathbf{f}}, \mu) & = \int_X (\underline{\mathbf{f}})_E d\mu & = \int_X \underline{\mathbf{f}_E} d\mu \\ & = V^l_X((\underline{\mathbf{f}})_E, \mu_\uparrow) & = V^l_X(\underline{\mathbf{f}_E}, \mu_\uparrow) \\ & = V^l_D((\underline{\mathbf{f}})_D, \mu_e) & = V^l_D(\underline{\mathbf{f}_D}, \mu_e) \\ & = \int_D (\underline{\mathbf{f}})_D d\overline{\mu}_e & = \int_D \underline{\mathbf{f}_D} d\overline{\mu}_e \end{array}$$

and

$$\begin{array}{lll} R^u_X(\overline{\mathbf{f}}, \mu) & = \int_X (\overline{\mathbf{f}})^E d\mu & = \int_X \overline{\mathbf{f}_E} d\mu \\ & = V^u_X((\overline{\mathbf{f}})^E, \mu_\uparrow) & = V^u_X(\overline{\mathbf{f}_E}, \mu_\uparrow) \\ & = V^u_D((\overline{\mathbf{f}})^D, \mu_e) & = V^u_D(\overline{\mathbf{f}_D}, \mu_e) \\ & = \int_D (\overline{\mathbf{f}})^D d\overline{\mu}_e & = \int_D \overline{\mathbf{f}_D} d\overline{\mu}_e. \end{array}$$

The conclusion follows from the definitions of the interval integrals. ■

In fact, we can drop the requirement of the embedding e being dense, if we define

$$V(\mathbf{f}_D, \nu) = \begin{cases} [V^l(\underline{\mathbf{f}}_D, \nu), V^u(\overline{\mathbf{f}}^D, \nu)] & \text{if } V^l(\underline{\mathbf{f}}_D, \nu) \leq V^u(\overline{\mathbf{f}}^D, \nu) \\ [-M_{\mathbf{f}}, M_{\mathbf{f}}] & \text{otherwise} \end{cases}$$

If $R(\mathbf{f}, \mu)$ is a singleton, then for every $f : X \to \mathbb{R}$ such that $f(x) \in \mathbf{f}(x)$ for all $x \in X$ (i.e., $\mathbf{f} \sqsubseteq f$) we have that f is continuous μ-almost everywhere.

8.5 Maximal Scott continuous functions in $[\mathbf{I}\overline{\mathbb{R}} \to \mathbf{I}\overline{\mathbb{R}}]$

Our aim in this section is to characterise the maximal Scott continuous functions from $\mathbf{I}\overline{\mathbb{R}}$ to $\mathbf{I}\overline{\mathbb{R}}$; that is, $\max[\mathbf{I}\overline{\mathbb{R}} \to \mathbf{I}\overline{\mathbb{R}}]$. First we show that $\max[\mathbf{I}\overline{\mathbb{R}} \to \mathbf{I}\overline{\mathbb{R}}] \simeq \max[\overline{\mathbb{R}} \to \mathbf{I}\overline{\mathbb{R}}]$, so it suffices to characterise the maximal Scott continuous functions in $[\overline{\mathbb{R}} \to \mathbf{I}\overline{\mathbb{R}}]$. A similar characterisation for a more restricted setting can be found in Section 2.4 of [AM96].

CONVENTION For simplicity for the rest of the chapter we make no distinction between $r \in \overline{\mathbb{R}}$ and the interval $[r, r] = \{r\}$. Another way of seeing this is to consider the set $\mathbf{I}\mathbb{R}\backslash\{\{x\} \mid x \in \overline{\mathbb{R}}\} \cup \overline{\mathbb{R}}$ instead of $\mathbf{I}\mathbb{R}$. Obviously for $\mathbf{z} \in \mathbf{I}\overline{\mathbb{R}}$ and $x \in \overline{\mathbb{R}}$ we have that $\mathbf{z} \sqsubseteq x$ if and only if $x \in \mathbf{z}$.

For $F : \mathbf{I}\overline{\mathbb{R}} \to \mathbf{I}\overline{\mathbb{R}}$ we define the **restriction** $F_r : \mathbb{R} \to \mathbf{I}\overline{\mathbb{R}}$ as

$$F_r(x) := F(x) \text{ for all } x \in \mathbb{R}.$$

Remark 8.12 *Suppose that* $F, G : \mathbf{I}\overline{\mathbb{R}} \to \mathbf{I}\overline{\mathbb{R}}$*. Then*

(a) If $F \sqsubseteq G$*, then* $F_r \sqsubseteq G_r$*.*

(b) If F *is continuous, then* F_r *is continuous.*

In light of the previous remark it makes sense to define $res : [\mathbf{I}\overline{\mathbb{R}} \to \mathbf{I}\overline{\mathbb{R}}] \to [\overline{\mathbb{R}} \to \mathbf{I}\overline{\mathbb{R}}]$ as $res(F) := F_r$.

Remark 8.13 *The operator res is not injective. Consider the functions* $F, G \in [\mathbf{I}\overline{\mathbb{R}} \to \mathbf{I}\overline{\mathbb{R}}]$ *given by*

$$F(\mathbf{z}) = \begin{cases} -1 & \textit{if } \overline{\mathbf{z}} < 0 \\ 1 & \textit{if } 0 < \underline{\mathbf{z}} \\ [-\frac{|\mathbf{z}|}{2}, \frac{|\mathbf{z}|}{2}] & \textit{if } 0 \in \mathbf{z} \textit{ and } |\mathbf{z}| \geq 2 \\ [-1, 1] & \textit{otherwise} \end{cases}$$
$$G(\mathbf{z}) = \begin{cases} -1 & \textit{if } \overline{\mathbf{z}} < 0 \\ 1 & \textit{if } 0 < \underline{\mathbf{z}} \\ [-1, 1] & \textit{otherwise} \end{cases}$$

then

$$F_r(x) = G_r(x) = \begin{cases} 1 & \textit{if } x < 0 \\ -1 & \textit{if } x > 0 \\ [-1, 1] & \textit{otherwise} \end{cases}$$

For $\mathbf{f} : \overline{\mathbb{R}} \to \mathbf{I}\overline{\mathbb{R}}$ define the **maximal monotone extension** $\mathbf{f}_m : \mathbf{I}\overline{\mathbb{R}} \to \mathbf{I}\overline{\mathbb{R}}$ as

$$\begin{aligned} \mathbf{f}_m(\mathbf{z}) &:= \bigwedge\{\mathbf{f}(x) \mid \mathbf{z} \sqsubseteq x\} \\ &= \bigwedge \mathbf{f}[\mathbf{z}] \\ &= [\inf \underline{\mathbf{f}}[\mathbf{z}], \sup \overline{\mathbf{f}}[\mathbf{z}]] \end{aligned}$$

Lemma 8.14 *Suppose that* $\mathbf{f}, \mathbf{g} : \overline{\mathbb{R}} \to \mathbf{I}\overline{\mathbb{R}}$*. Then*

(a) $\mathbf{f}_m$ *extends* $\mathbf{f}$. *That is,* $\mathbf{f}_m(x) = \mathbf{f}(x)$ *for all* $x \in \mathbb{R}$.

(b) $\mathbf{f}_m$ *is monotone.*

(c) $\mathbf{f} = (\mathbf{f}_m)_r$

(d) *If* $F : \mathbf{I}\overline{\mathbb{R}} \to \mathbf{I}\overline{\mathbb{R}}$ *is monotone and extends* $\mathbf{f}$, *then* $F \sqsubseteq \mathbf{f}_m$.

(e) $\mathbf{f} \sqsubseteq \mathbf{g}$ *if and only if* $\mathbf{f}_m \sqsubseteq \mathbf{g}_m$.

(f) *If* $\mathbf{f} \neq \mathbf{g}$, *then* $\mathbf{f}_m \neq \mathbf{g}_m$.

(g) $\mathbf{f}$ *is continuous if and only if* $\mathbf{f}_m$ *is continuous.*

Proof

(a) $\mathbf{f}_m(x) = \bigwedge\{\mathbf{f}(x)\}$ by definition.

(b) If $\mathbf{x} \sqsubseteq \mathbf{z}$, then $\mathbf{f}[\mathbf{z}] \subseteq \mathbf{f}[\mathbf{x}]$. Therefore $\mathbf{f}_m(\mathbf{x}) = \bigwedge \mathbf{f}[\mathbf{x}] \sqsubseteq \bigwedge \mathbf{f}[\mathbf{z}] = \mathbf{f}_m(\mathbf{z})$.

(c) Is immediate from the definitions and part (a).

(d) Suppose that $\mathbf{z} \in \mathbf{I}\overline{\mathbb{R}}$. For all $x \in \mathbf{z}$ we have that $\mathbf{z} \sqsubseteq x$ hence $F(\mathbf{z}) \sqsubseteq F(x) = \mathbf{f}(x)$ since F is monotone and extends $\mathbf{f}$. Therefore $F(\mathbf{z}) \sqsubseteq \bigwedge\{\mathbf{f}(x) \mid \mathbf{z} \sqsubseteq x\} = \mathbf{f}_m(\mathbf{z})$.

(e) $\boxed{\Leftarrow}$ Follows from part (a). $\boxed{\Rightarrow}$ $\mathbf{f}_m(\mathbf{z}) = \bigwedge\{\mathbf{f}(x) \mid \mathbf{z} \sqsubseteq x\} \sqsubseteq \bigwedge\{\mathbf{g}(x) \mid \mathbf{z} \sqsubseteq x\} = \mathbf{g}_m(\mathbf{z})$.

(f) If $\mathbf{f}(x) \neq \mathbf{g}(x)$, then $\mathbf{f}_m(x) \neq \mathbf{g}_m(x)$.

(g) $\boxed{\Leftarrow}$ Follows immediately from part (c) and Remark 8.12-(b). $\boxed{\Rightarrow}$ Suppose that $\mathbf{x} \in \mathbf{I}\overline{\mathbb{R}}$ and $\mathbf{z} \ll \mathbf{f}_m(\mathbf{x})$. Then for all $y \in \mathbf{x}$ we have that $\mathbf{x} \sqsubseteq y$, hence $\mathbf{z} \ll \mathbf{f}_m(\mathbf{x}) \sqsubseteq \mathbf{f}_m(y) = \mathbf{f}(y)$ by parts (b) and (a). Since $\mathbf{f}$ is a continuous function then $G := \mathbf{f}^{-1}(\twoheaduparrow\mathbf{z})$ is an open subset of $\overline{\mathbb{R}}$ such that $\mathbf{x} \subseteq G$. Thus, $[\underline{G}, \overline{G}] \ll \mathbf{x}$ and for every $\mathbf{x}'$ such that $[\underline{G}, \overline{G}] \ll \mathbf{x}'$ we have that $\mathbf{z} \ll \mathbf{f}(y)$ for all $y \in \mathbf{x}'$. This implies that $\mathbf{z} \sqsubseteq \mathbf{f}_m(\mathbf{x}')$. Summarising, we have shown that if $\mathbf{x} \in \mathbf{I}\overline{\mathbb{R}}$ and $\mathbf{z} \ll \mathbf{f}_m(\mathbf{x})$, then there exists $[\underline{G}, \overline{G}] \in \mathbf{I}\overline{\mathbb{R}}$ with $[\underline{G}, \overline{G}] \ll \mathbf{x}$ such that for all $\mathbf{x}' \in \mathbf{I}\overline{\mathbb{R}}$

$$[\underline{G}, \overline{G}] \ll \mathbf{x}' \text{ implies that } \mathbf{z} \sqsubseteq \mathbf{f}_m(\mathbf{x}').$$

Using the interpolation property (Proposition 1.3) it is easy to strengthen the last inequality to $\mathbf{z} \ll \mathbf{f}_m(\mathbf{x}')$. Since $\mathbf{I}\overline{\mathbb{R}}$ is a continuous dcpo, this shows that $\mathbf{f}_m$ is continuous at $\mathbf{x}$. ■

We define $mon : [\overline{\mathbb{R}} \to \mathbf{I}\overline{\mathbb{R}}] \to [\mathbf{I}\overline{\mathbb{R}} \to \mathbf{I}\overline{\mathbb{R}}]$ as $mon(\mathbf{f}) := \mathbf{f}_m$.

Lemma 8.15 *Let* $F : \mathbf{I}\overline{\mathbb{R}} \to \mathbf{I}\overline{\mathbb{R}}$ *be a monotone function. Then*

(a) $F \sqsubseteq (F_r)_m$. *Hence,* $(F_r)_m$ *is the greatest monotone function that agrees with* F *on* $\max(\mathbf{I}\overline{\mathbb{R}}) = \overline{\mathbb{R}}$.

(b) *If* $F \in \max[\mathbf{I}\overline{\mathbb{R}} \to \mathbf{I}\overline{\mathbb{R}}]$, *then* $F_r \in \max[\overline{\mathbb{R}} \to \mathbf{I}\overline{\mathbb{R}}]$.

Proof

(a) By definition F extends F_r. By Lemma 8.14-(d) it follows that $F \sqsubseteq (F_r)_m$.

(b) Suppose that $\mathbf{g} \in [\overline{\mathbb{R}} \to \mathbf{I}\overline{\mathbb{R}}]$ and $F_r \sqsubseteq \mathbf{g}$, then by Lemma 8.14-(g) $\mathbf{g}_m \in [\mathbf{I}\overline{\mathbb{R}} \to \mathbf{I}\overline{\mathbb{R}}]$. By part (a) and Lemma 8.14-(e) we have that $F \sqsubseteq (F_r)_m \sqsubseteq \mathbf{g}_m$. But F is maximal on $[\mathbf{I}\overline{\mathbb{R}} \to \mathbf{I}\overline{\mathbb{R}}]$, therefore $F = \mathbf{g}_m$. By Lemma 8.14-(c) we have that $F_r = (\mathbf{g}_m)_r = \mathbf{g}$. Hence F_r is maximal. ■

For the functions defined in Remark 8.13 we have that F is strictly below $(F_r)_m = G$.

Corollary 8.16 (res, mon) *is an adjunction between* $[\mathbf{I}\overline{\mathbb{R}} \to \mathbf{I}\overline{\mathbb{R}}]$ *and* $[\overline{\mathbb{R}} \to \mathbf{I}\overline{\mathbb{R}}]$.

Proof By Lemma 8.14-(c) we have that $res \circ mon = Id_{[\overline{\mathbb{R}} \to \mathbf{I}\overline{\mathbb{R}}]}$ and by Lemma 8.15-(a) we know that $mon \circ res \sqsupseteq Id_{[\overline{\mathbb{R}} \to \mathbf{I}\overline{\mathbb{R}}]}$. The conclusion follows from part (3) of Proposition 3.1.10 in [AJ94].

Theorem 8.17 ***First Characterization Theorem***

1. $\mathbf{f} \in \max[\overline{\mathbb{R}} \to \mathbf{I}\overline{\mathbb{R}}]$ *if and only if* $\mathbf{f}_m \in \max[\mathbf{I}\overline{\mathbb{R}} \to \mathbf{I}\overline{\mathbb{R}}]$ *(recall that* $\mathbf{f} = (\mathbf{f}_m)_r$*).*

2. $F \in \max[\mathbf{I}\overline{\mathbb{R}} \to \mathbf{I}\overline{\mathbb{R}}]$ *if and only if* $F_r \in \max[\overline{\mathbb{R}} \to \mathbf{I}\overline{\mathbb{R}}]$ *and* $F = (F_r)_m$.

That is, $\max[\mathbf{I}\overline{\mathbb{R}} \to \mathbf{I}\overline{\mathbb{R}}]$ *and* $\max[\overline{\mathbb{R}} \to \mathbf{I}\overline{\mathbb{R}}]$ *are order isomorphic.*

Proof

1. $\boxed{\Rightarrow}$ Suppose that $G \in [\mathbf{I}\overline{\mathbb{R}} \to \mathbf{I}\overline{\mathbb{R}}]$ is such that $\mathbf{f}_m \sqsubseteq G$. By Lemma 8.14-(c) and Remark 8.12-(a) we have that $\mathbf{f} = (\mathbf{f}_m)_r \sqsubseteq G_r$. Hence $\mathbf{f} = G_r$ since $\mathbf{f}$ is maximal. By Lemma 8.15-(a) we get $G \sqsubseteq (G_r)_m = \mathbf{f}_m$. Therefore $\mathbf{f}_m = G$. $\boxed{\Leftarrow}$ Since $\mathbf{f}_m$ is maximal, then by Lemma 8.15-(b) we have that $\mathbf{f} = (\mathbf{f}_m)_r \in \max[\overline{\mathbb{R}} \to \mathbf{I}\overline{\mathbb{R}}]$.

2. $\boxed{\Rightarrow}$By Lemma 8.15 we know that $F_r \in \max[\overline{\mathbb{R}} \to \mathbf{I}\overline{\mathbb{R}}]$ and $F \sqsubseteq (F_r)_m$. But F is maximal, therefore $F = (F_r)_m$. $\boxed{\Leftarrow}$By part (1) $F = (F_r)_m \in \max[\mathbf{I}\overline{\mathbb{R}} \to \mathbf{I}\overline{\mathbb{R}}]$.

We know that mon and res are order preserving and monotone. Since $mon \circ res$ is the identity on $\max[\mathbf{I}\overline{\mathbb{R}} \to \mathbf{I}\overline{\mathbb{R}}]$ and $res \circ mon$ is the identity on $\max[\overline{\mathbb{R}} \to \mathbf{I}\overline{\mathbb{R}}]$ it follows that $res : \max[\mathbf{I}\overline{\mathbb{R}} \to \mathbf{I}\overline{\mathbb{R}}] \to \max[\overline{\mathbb{R}} \to \mathbf{I}\overline{\mathbb{R}}]$ and $mon : \max[\overline{\mathbb{R}} \to \mathbf{I}\overline{\mathbb{R}}] \to \max[\mathbf{I}\overline{\mathbb{R}} \to \mathbf{I}\overline{\mathbb{R}}]$ are bijective. ■

In the light of the previous theorem we focus our attention on functions in $[\overline{\mathbb{R}} \to \mathbf{I}\overline{\mathbb{R}}]$. Suppose that $(Y, \mathcal{H})$ is a T_0 topological space. For $\mathbf{f} : Y \to \mathbf{I}\overline{\mathbb{R}}$ we have that

$$S_{\mathbf{f}} = \{x \in Y \mid \mathbf{f} \text{ is continuous on } x \text{ and } \exists y \in \overline{\mathbb{R}}.\mathbf{f}(x) = y\}$$

As a consequence of Lemma 8.7-(6) we know that if $S_{\mathbf{f}}$ is dense in Y, then the function

$$\begin{aligned} \mathbf{f}_c(y) &= \bigsqcup{}^{\uparrow}\{\bigwedge \mathbf{f}[G \cap S_{\mathbf{f}}] \mid G \in \mathcal{N}(y)\} \\ &= \bigwedge\{\underline{\lim}_j \, \mathbf{f}(x_j) \mid (x_j)_{j \in J} \text{ net in } S_f \text{ with } (x_j) \to y\} \end{aligned}$$

is continuous and maximal. That is, $\mathbf{f}_c \in \max[Y \to \mathbf{I}\overline{\mathbb{R}}]$. It is also useful to have the following result.

Corollary 8.18 *Suppose that $(Y, \mathcal{H})$ is a T_0 topological space. Let $\mathbf{f} : Y \to \mathbf{I}\overline{\mathbb{R}}$ be any function such that $S_{\mathbf{f}}$ is dense in Y. Then for all $y \in Y$, there exist nets $(l_j)_{j \in J}$ and $(r_j)_{j \in J}$ in $S_{\mathbf{f}}$ such that*

1. *$(r_j) \to y$ and $\overline{\mathbf{f}_c}(y) = \lim_j \overline{\mathbf{f}}(r_j) = \lim_j \underline{\mathbf{f}}(r_j)$.*
2. *$(l_k) \to y$ and $\underline{\mathbf{f}_c}(y) = \lim_j \underline{\mathbf{f}}(l_j) = \lim_j \overline{\mathbf{f}}(l_j)$.*

Proof Fix $y \in Y$. Let $\mathbf{k}_G := \bigwedge \mathbf{f}[G \cap S_{\mathbf{f}}]$ for every $G \in \mathcal{N}(y)$. Then there exists sequences $(l_{(G,n)})_{n \in \mathbb{N}}$ and $(r_{(G,n)})_{n \in \mathbb{N}}$ in $G \cap S_{\mathbf{f}}$ such that

$$\begin{matrix} \overline{\mathbf{k}_G} = \lim_n \overline{\mathbf{f}}(r_{(G,n)}) \\ \underline{\mathbf{k}_G} = \lim_n \underline{\mathbf{f}}(l_{(G,n)}) \end{matrix} \quad \text{(the limits are taken in } \mathbb{R}^{\lambda}\text{)}$$

Let $J := \mathcal{N}(y) \times \mathbb{N}$ ordered lexicographically; that is, $(G, n) \leq (H, m)$ if $G \supseteq H$ or if ($G = H$ and $n \leq m$). The rest of the proof follows as in part (5) of Lemma 8.7. ■

Since $\overline{\mathbb{R}}$ with its metric topology is a second countable space, the following result is also easily verified.

Corollary 8.19 *Let $\mathbf{f} : \overline{\mathbb{R}} \to \mathbf{I}\overline{\mathbb{R}}$ be any function such that $S_{\mathbf{f}}$ is dense in $\overline{\mathbb{R}}$. Then for all $y \in \overline{\mathbb{R}}$, there exists sequences $(l_n)_{n \in \mathbb{N}}$ and $(r_n)_{n \in \mathbb{N}}$ in $S_{\mathbf{f}} \backslash \{y\}$ such that*

1. *$(r_n) \to y$ and $\overline{\mathbf{f}_c}(y) = \lim_n \overline{\mathbf{f}}(r_n) = \lim_n \underline{\mathbf{f}}(r_n)$.*

2. $(l_n) \to y$ *and* $\underline{\mathbf{f}_c}(y) = \lim_n \underline{\mathbf{f}}(l_n) = \lim_n \overline{\mathbf{f}}(l_n)$.

Remark 8.20 *Recall from the theory of real analysis that if a function* $f : \mathbb{R} \to \mathbb{R}$ *is bounded on an open interval* E *then for all* $T \subseteq E$ *we define the oscillation of* f *on* T *as* $\Omega_f(T) = \sup\{f(x) - f(y) \mid x, y \in T\}$. *For all* $x \in E$ *the oscillation of* f *at* x *is defined as* $\omega_f(x) = \lim_{h \to 0^+} \Omega_f(E \cap (x-h, x+h))$. *Is not difficult to see that* $|f_c(x)| = \omega_f(x)$.

Example 8.21 *Con removes non-jump (removable) discontinuities. Suppose that* $\mathbf{f} : \mathbb{R} \to \mathbf{I}\mathbb{R}$ *is given by*

$$\mathbf{f}(x) := \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

Then $\mathbf{f}_c(x) = 0$ *for all* x.

Example 8.22 *Similar to the previous example but with* $\mathbf{f}$ *continuous. Suppose that* $\mathbf{f} : \mathbb{R} \to \mathbf{I}\mathbb{R}$ *is given by*

$$\mathbf{f}(x) := \begin{cases} [0, \frac{1}{q}] & \text{if } x = p/q \in \mathbb{Q} \text{ and } (p,q) = 1 \text{ (i.e. } p, q \text{ relatively prime)} \\ 0 & \text{if } x \in \mathbb{R} \backslash \mathbb{Q} \end{cases}$$

Then $\mathbf{f}_c(x) = 0$ *for all* x.

Example 8.23 *Con minimizes intervals. Suppose that* $\mathbf{f} : \mathbb{R} \to \mathbf{I}\mathbb{R}$ *is given by*

$$\mathbf{f}(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \\ [0,2] & \text{if } x = 0 \end{cases}$$

then

$$\mathbf{f}_c(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \\ [0,1] & \text{if } x = 0 \end{cases}$$

For the rest of the section $(Y, \mathcal{H})$ denotes a topological space. A subset $N \subseteq Y$ is **nowhere dense** if $int(cl(N)) = \emptyset$. That is, the complement of $cl(N)$ is a dense subset of Y. Obviously, a closed set F is nowhere dense if and only if $int(F) = \emptyset$. The space Y is called a **Baire space** if the countable union of nowhere dense sets has empty interior.

Proposition 8.24 *(Baire) Locally compact Hausdorff spaces are Baire spaces.*

The next result shows that for a broad class of spaces the functions of the form $\mathbf{f}_c$ are the only continuous maximal functions.

Theorem 8.25 ***Second Characterization Theorem***

Suppose that $(Y, \mathcal{H})$ is a T_0 regular Baire space and $\mathbf{f} : Y \to \mathbf{I}\overline{\mathbb{R}}$ is continuous. Then $\mathbf{f}$ is maximal if and only if the following two conditions hold:

1. $S_{\mathbf{f}} = \{x \mid \mathbf{f}(x) \in \overline{\mathbb{R}}\}$ *is a dense subset of* Y.
2. $\mathbf{f} = \mathbf{f}_c$.

Proof

$\boxed{\Leftarrow}$ Immediate from part (3) of Lemma 8.7.

$\boxed{\Rightarrow}$ Suppose that $\mathbf{f} : Y \to \mathbf{I}\overline{\mathbb{R}}$ is continuous and $M := S_{\mathbf{f}}$ is not dense in Y. We build a continuous function $\mathbf{g}$ strictly greater than $\mathbf{f}$. Since M is not dense, there exists an open set $G_1 \neq \emptyset$ such that $G_1 \subseteq M^{\mathsf{c}}$. Hence for all $y \in G_1$ we have that $|\mathbf{f}(y)| > 0$. For every $n \in \mathbb{N}$, let $J_n := cl(\{y \in G_1 \mid |\mathbf{f}(y)| \geq \frac{1}{n}\})$. Then (J_n) is a sequence of closed sets in Y such that $G_1 \subseteq \bigcup_{n \in \mathbb{N}} J_n$.

1. Claim: $|\mathbf{f}(y)| \geq \frac{1}{n}$ for all $y \in J_n$. Suppose that $y \in J_n$. Then by part (1) of Proposition 1.9, there exists a net $(y_j)_{j \in J}$ in G_1 with $(y_j) \to y$ and $|\mathbf{f}(y_j)| \geq \frac{1}{n}$ for all $j \in J$. Since the functions $\mathbf{f} : Y \to \mathbf{I}\overline{\mathbb{R}}$ and $|\_| : \mathbf{I}\overline{\mathbb{R}} \to [0, \infty]^{\geq}$ are continuous then the function $|\mathbf{f}| := |\_| \circ \mathbf{f}$ is continuous. By Corollary 8.2 it follows that $|\mathbf{f}(y)| \sqsubseteq \liminf_j |\mathbf{f}(y_j)| \sqsubseteq \frac{1}{n}$ in $([0, \infty], \geq)$. That is, $|\mathbf{f}(y)| \geq \liminf_j |\mathbf{f}(y_j)| \geq \frac{1}{n}$.

2. Claim: there exists $m \in \mathbb{N}$ and an open set $G_2 \in \mathcal{H}$ such that $|\mathbf{f}(y)| \geq \frac{1}{m}$ for all $y \in G_2$. If J_n has empty interior, then J_n is nowhere dense since it is closed. Since Y is a Baire space by assumption, if $int(J_n) = \emptyset$ for all $n \in \mathbb{N}$, then $\bigcup_{n \in \mathbb{N}} J_n$ would have empty interior, which is false. Therefore there must exist $m \in \mathbb{N}$ such that $int(J_m) \neq \emptyset$. That is, there exists a nonempty open set $G_2 \subseteq J_m$. Hence $|f(x)| \geq \frac{1}{m}$ for all $x \in G_2$.

3. Claim: there exists an open set $G_3 \in \mathcal{H}$ and $t \in \overline{\mathbb{R}}$ such that $t \in \mathbf{f}(y)$ for all $y \in G_3$. Let $u = \inf_{y \in G_2}\{|\mathbf{f}(y)|\}$.

- **Case** $u = \infty$. If $\mathbf{f}(y) = [-\infty, \infty]$ for all $y \in G_2$. Let $G_3 := G_2$ and $t = 0$. Otherwise, there exists $z \in G_2$ such that $\mathbf{f}(z) = [a, b]$ with $a \neq -\infty$ or $b \neq \infty$. Without loss of generality assume that $a \neq -\infty$. Then $b = \infty$ since $\mathbf{f}(z)$ has infinite length. Choose any $c \in \mathbb{R}$ with $-\infty < c < a$. Since $\mathbf{f}$ is continuous we can find an open set $G_3 \in \mathcal{N}(z)$ with $G_3 \subseteq G_2$ such that $[c, \infty] \ll \mathbf{f}(y)$ for all $y \in G_3$. Moreover, since $G_3 \subseteq G_2$ we have that $|\mathbf{f}(y)| = u = \infty$ for all $y \in G_3$. This means that $\infty \in \mathbf{f}(y)$ for all $y \in G_3$. Let $t = \infty$.

- **Case** $u \in \mathbb{R}$. Find $z \in G_2$ such that $u \leq |\mathbf{f}(z)| < \frac{10}{8}u$. Since $\mathbf{f}$ is continuous we can find an open set $G_3 \in \mathcal{N}(z)$ with $G_3 \subseteq G_2$ such that $[\underline{\mathbf{f}(z)} - \frac{1}{8}u, \overline{\mathbf{f}(z)} + \frac{1}{8}u] \ll \mathbf{f}(y)$ for all $y \in G_3$. Let t be the midpoint of the interval $\mathbf{f}(z)$, we claim that $t \in \mathbf{f}(y)$ for all $y \in G_3$. The reason is the following:

$$\begin{array}{llll} & u & \leq |\mathbf{f}(z)| & < \frac{10}{8}u \\ \Rightarrow & t + \frac{4}{8}u & \leq \overline{\mathbf{f}(z)} & < t + \frac{5}{8}u \\ \Rightarrow & & \overline{\mathbf{f}(y)} & < t + \frac{6}{8}u \\ \Rightarrow & \underline{\mathbf{f}(y)} & \leq \overline{\mathbf{f}(y)} - u & < t - \frac{2}{8}u \end{array}$$

The last implication holds since $|\mathbf{f}(y)| \geq u$ for all $y \in G_3$. The symmetric argument shows that

$$\underline{\mathbf{f}(y)} < t - \frac{2}{8}u < t + \frac{2}{8}u < \overline{\mathbf{f}(y)}$$

Since Y is regular by assumption then we can find an open set $G_4 \in \mathcal{H}$ such that $z \in G_4 \subseteq cl(G_4) \subseteq G_3$. Define the $\mathbf{g} : Y \to \mathbf{I}\overline{\mathbb{R}}$ as:

$$\mathbf{g}(x) = \begin{cases} t & \text{if } x \in G_4 \\ \mathbf{f}(x) & \text{otherwise} \end{cases}$$

Clearly $\mathbf{f} \sqsubseteq \mathbf{g}$ since $t \in \mathbf{f}(x)$ for all $x \in G_4$. Clearly they are not equal since $\mathbf{f}(z)$ is not maximal while $\mathbf{g}(z) = t$. Function $\mathbf{g}$ is obviously continuous on G_4 since it is constant, and $\mathbf{g}$ is continuous on $(cl(G_4))^{\mathsf{c}}$ since it agrees with $\mathbf{f}$. Therefore for establishing the continuity of $\mathbf{g}$ we only have to check that $\mathbf{g}$ is continuous on ∂G_4.

Take any point $w \in \partial G_4$ and suppose that $\mathbf{x} \ll \mathbf{g}(w) = \mathbf{f}(w)$. Since $\mathbf{f}$ is continuous there exists an open set $G \in \mathcal{N}(w)$ such that $\mathbf{x} \ll \mathbf{f}(y)$ for all $y \in G$. Notice that $G = (G \cap G_4) \cup (G \backslash G_4)$. If $y \in (G \backslash G_4)$, then $\mathbf{x} \ll \mathbf{f}(y) = \mathbf{g}(y)$. If $y \in (G \cap G_4)$, then $\mathbf{g}(y) = t$. But recall that $w \in \partial G_4 \subseteq cl(G_4) \subseteq G_3$. Hence $t \in \mathbf{f}(w)$ by Claim 3. Thus $\mathbf{x} \ll \mathbf{f}(w) \sqsubseteq t = \mathbf{g}(y)$. Summarising, we have shown that $\mathbf{x} \ll \mathbf{g}(y)$ for all $y \in G$. Which shows that $\mathbf{g}$ is continuous at w, hence $\mathbf{g}$ is continuous on ∂G_4.

We have shown that $S_{\mathbf{f}}$ not dense in Y implies that $\mathbf{f}$ is not maximal, which proves condition (1). Since $\mathbf{f}$ is continuous and we have shown that $S_{\mathbf{f}}$ is dense on Y, then by Lemma 8.7-(4) we have that $\mathbf{f} \sqsubseteq \mathbf{f}_c$. But $\mathbf{f}$ is maximal, therefore condition (2) holds. ■

Remark 8.26 *The function* $\mathbf{f}$ *in Example 8.23 shows that the density of* $S_{\mathbf{f}}$ *is not enough by itself to guarantee that* $\mathbf{f}$ *is maximal. Hence, condition (1) in the previous theorem does not imply that* $\mathbf{f}$ *is maximal. Similarly* $\mathbf{f}_c$

cannot be defined unless $S_{\mathbf{f}}$ is dense. Hence condition (2) by itself does not have any meaning.

J. R. Isbell [Isb75a, 4.2] has shown that locally compact sober spaces are Baire spaces (see also [GHK$^+$80, Corollary I-3.43.10]). Locally compact sober spaces need not be regular. Hence the following should be an interesting question:

Problem 8.27 *Does Theorem 8.25 hold when $(Y, \mathcal{H})$ is a locally compact sober space?*

Corollary 8.28 *Let $\mathbf{f} : \overline{\mathbb{R}} \to \mathbf{I}\overline{\mathbb{R}}$ be continuous. Then $\mathbf{f}$ is maximal if and only if the following two conditions hold:*

1. *$S_{\mathbf{f}} = \{x \mid \mathbf{f}(x) \in \overline{\mathbb{R}}\}$ is a dense subset of $\overline{\mathbb{R}}$.*
2. *$\mathbf{f} = \mathbf{f}_c$.*

¿From Theorem 8.17 we immediately get.

Corollary 8.29 *Let $F : \mathbf{I}\overline{\mathbb{R}} \to \mathbf{I}\overline{\mathbb{R}}$ be continuous. Then F is maximal if and only if the following two conditions hold:*

1. *$S_{F_r} = \{x \in \overline{\mathbb{R}} \mid F(x) \in \overline{\mathbb{R}}\}$ is a dense subset of $\overline{\mathbb{R}}$.*
2. *$F = ((F_r)_c)_m = mon(Con(res(F)))$.*

Corollary 8.30 *Let $F : \mathbf{I}\overline{\mathbb{R}} \to \mathbf{I}\overline{\mathbb{R}}$ be continuous. Then F is maximal if and only if the following conditions hold:*

1. *$S_{F_r} = \{x \in \overline{\mathbb{R}} \mid F(x) \in \overline{\mathbb{R}}\}$ is a dense subset of $\overline{\mathbb{R}}$.*
2. *$F = (F_r)_m$.*
3. *For all $x \in \overline{\mathbb{R}}$ there exist sequences (y_n) and (z_n) in $S_{F_r} \backslash \{x\}$ converging to x and such that*

$$F(x) = [\lim_n \underline{F(y_n)}, \lim_n \overline{F(z_n)}]$$

Proof Notice that Condition (3) is equivalent to:

4. For all $x \in \overline{\mathbb{R}}$ there exist sequences (y_n) and (z_n) in $S_{F_r} \backslash \{x\}$ converging to x and such that

   $F_r(x) = [\lim_n \underline{F_r(y_n)}, \lim_n \overline{F_r(z_n)}]$.

$\boxed{\Rightarrow}$ If F is maximal, then by Theorem 8.17 $F = (F_r)_m$ and F_r is maximal. Then by Corollary 8.29 S_{F_r} is dense and $F_r = (F_r)_c$. From the latter equality and Corollary 8.19 we recover Condition (4). $\boxed{\Leftarrow}$ By Lemma 8.7-(6) $F_r(x) = [\lim_n \underline{F_r(y_n)}, \lim_n \overline{F_r(z_n)}]$ implies that $(F_r)_c \sqsubseteq F_r$. By Remark 8.12-(b) F_r is continuous. Hence by parts (4) and (3) of Lemma 8.7 $F_r = (F_r)_c$ and F_r is maximal. Thus, F is maximal by Theorem 8.17. ■

8.6 Representation and refinement of real functions

Suppose that $f : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ and $F : \mathbf{I}\overline{\mathbb{R}} \to \mathbf{I}\overline{\mathbb{R}}$. We say that f **refines** F if $F \sqsubseteq f$.

Corollary 8.31 *Suppose that* $F \in \max[\mathbf{I}\overline{\mathbb{R}} \to \mathbf{I}\overline{\mathbb{R}}]$ *and* $f : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ *is a refinement of* F. *Then*

(a) f is continuous at x if and only if $F(x)$ is maximal. That is, $C_f = S_{F_r}$.

(b) C_f is dense in $\overline{\mathbb{R}}$.

Proof By Theorem 8.17 we have that $F_r \in \max[\overline{\mathbb{R}} \to \mathbf{I}\overline{\mathbb{R}}]$. Hence Theorem 8.25 implies that S_{F_r} is dense in $\overline{\mathbb{R}}$ and $F_r = (F_r)_c$. Take $x \in S_{F_r}$, then F_r is continuous at x and $F_r(x) \in \overline{\mathbb{R}}$. Since $F(x) = F_r(x) \sqsubseteq f(x)$, it follows that $F_r(x) = f(x)$.

(a) $\boxed{\supseteq}$ Suppose that $x \in S_{F_r}$ and $\mathbf{z} \ll F_r(x) = f(x)$. Since F_r is continuous at x, there exists an open set $H \in \mathcal{N}(x)$ such that $\mathbf{z} \ll F_r(y)$ for all $y \in H$. But $F_r \sqsubseteq f$, implies that $\mathbf{z} \ll f(y)$ for all $y \in H$. Which shows that f is continuous at x. Thus $x \in C_f$. $\boxed{\subseteq}$ Suppose that $x \in C_f$. By part (3) of Lemma 8.7, there exists a net $(x_j)_{j \in J}$ in S_{F_r} such that $(x_j) \to x$ and $(F_r)_c(x) = \underline{\lim}_j F_r(x_j) = \underline{\lim}_j f(x_j)$. Hence

$$\begin{aligned} F(x) &\equiv F_r(x) && \text{since } x \in \mathbb{R} \\ &= (F_r)_c(x) \\ &= \underline{\lim}_j F_r(x_j) \\ &= \underline{\lim}_j f(x_j) \\ &\sqsupseteq f(x) && \text{By 8.2 since } x \in C_f \end{aligned}$$

Since $f(x)$ is maximal then $F_r(x) = f(x)$. Hence $x \in S_{F_r}$.

(b) By part (a) we know that $C_f = S_{F_r}$ but we observed that S_{F_r} is dense in $\overline{\mathbb{R}}$. ■

Corollary 8.32 *Let* $f : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$. *Suppose that* C_f *is dense in* $\overline{\mathbb{R}}$, *let* $F := (f_c)_m$. *Then* $F \in \max[\mathbf{I}\overline{\mathbb{R}} \to \mathbf{I}\overline{\mathbb{R}}]$ *and* $F(x) = f(x)$ *for all* $x \in C_f$.

Proof Note that $S_f = C_f$, hence by Theorem 8.25 it follows that f_c is maximal. By Theorem 8.17 we have that $F = (f_c)_m \in \max[\mathbf{I}\overline{\mathbb{R}} \to \mathbf{I}\overline{\mathbb{R}}]$. If

$x \in C_f = S_f$, then by Lemmas 8.7-(1) and 8.14-(a) we have that $f(x) = f_c(x) = F(x)$. ■

Corollary 8.33 *Let $f : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$. Then the following conditions are equivalent:*

1. *There exists $F \in \max[\mathbf{I}\overline{\mathbb{R}} \to \mathbf{I}\overline{\mathbb{R}}]$ such that $F \sqsubseteq f$.*
2. *C_f is dense in $\overline{\mathbb{R}}$ and for all $x \in \overline{\mathbb{R}}$ there exist sequences (y_n) and (z_n) in $C_f \backslash \{x\}$ converging to x and such that*

$$\lim_n f(y_n) \leq f(x) \leq \lim_n f(z_n)$$

Proof $\boxed{1 \Rightarrow 2}$ Fix $x \in \overline{\mathbb{R}}$. By Theorems 8.17 and 8.25 we have that $F_r \in \max[\overline{\mathbb{R}} \to \mathbf{I}\overline{\mathbb{R}}]$, S_{F_r} is dense and $F_r = (F_r)_c$. By Corollary 8.19, there exist sequences (y_n) and (z_n) in $S_{F_r} \backslash \{x\}$ converging to x and such that $\overline{(F_r)_c(x)} = \lim_n \overline{F_r}(z_n)$ and $\underline{(F_r)_c(x)} = \lim_n \underline{F_r}(y_n)$. Notice that for all $z \in S_{F_r}$ we have that $f(z) = F(z) = F_r(z) \in \overline{\mathbb{R}}$. Hence

$$\begin{aligned} f(x) &\sqsupseteq F(x) \\ &= F_r(x) \\ &= (F_r)_c(x) \\ &= [\lim_n \underline{F_r}(y_n), \lim_n \overline{F_r}(z_n)] \\ &= [\lim_n f(y_n), \lim_n f(z_n)] \qquad \text{since } y_n,\ z_n \in S_{F_r} \end{aligned}$$

By Corollary 8.31 we know that $S_{F_r} = C_f$ and the claim follows.

$\boxed{2 \Rightarrow 1}$ By part (3) of Lemma 8.7 $f_c \in \max[\overline{\mathbb{R}} \to \mathbf{I}\overline{\mathbb{R}}]$. Let $F := (f_c)_m$, then F is maximal by Theorem 8.17, and for all $x \in \overline{\mathbb{R}}$

$$\begin{aligned} F(x) &= f_c(x) && \text{by 8.14-(a)} \\ &= \bigwedge\{\underline{\lim}_j f(x_j) \mid (x_j) \in Net(x)\} \\ &\sqsubseteq \lim_n f(y_n) \wedge \lim_n f(z_n) \\ &= [\lim_n f(y_n), \lim_n f(z_n)] && \text{since } \lim_n f(y_n) \leq \lim_n f(z_n) \\ &\sqsubseteq f(x) && \text{by assumption} \end{aligned}$$

■

Definition 8.34 *Suppose that $f : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ and $F : \mathbf{I}\overline{\mathbb{R}} \to \mathbf{I}\overline{\mathbb{R}}$.*

- *f is **representable** if there exists $F \in \max[\mathbf{I}\overline{\mathbb{R}} \to \mathbf{I}\overline{\mathbb{R}}]$ such that $F(x) = f(x)$ for all $x \in C_f$.*
- *f is **maximal extendable** if there exists $F \in \max[\mathbf{I}\overline{\mathbb{R}} \to \mathbf{I}\overline{\mathbb{R}}]$ such that $F(x) = f(x)$ for all $x \in \overline{\mathbb{R}}$.*

Observe that maximal extendable implies representable.

Example 8.35 *f representable but not maximal extendable.*

$$f(x) := \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$$
$$g(x) := \begin{cases} [0,1] & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$$
$$F(\mathbf{z}) := 0 \text{ for all } \mathbf{z} \in \mathbf{I}\overline{\mathbb{R}}$$

*$(g)_m$ represents f but is not maximal since $(g)_m \sqsubseteq F$. F is maximal and represents f but $F \sqsubseteq f$ does **not** hold.*

Let $\lambda : \mathcal{B}([0,1]) \to [0,1]$ be the Lebesgue measure.

Theorem 8.36 *If $f : \mathbb{R} \to \mathbb{R}$ is Riemann integrable then f is maximal extendable.*

Proof Let $D = (C_f)^{\mathsf{c}}$. If f is not maximal extendable, then C_f is not dense. Hence there exists a non empty $(a,b) \subseteq D$. Hence $\lambda(D) \geq b - a > 0$. Therefore f is not Riemann integrable. ■

For the following example we need to construct a Cantor set $\mathcal{C}$ such that $\lambda(\mathcal{C}) > 0$. We do this in the most effortless possible way. The following construction was also used by R. Heckmann in another context. Choose any sequence (a_n) in $(0,1)$ such that $\sum_{n=1}^{\infty} a_n < 1$. For example, $a_n = \frac{\varepsilon}{2^n}$ with $\varepsilon < 1$. For simplicity we define $a_0 := 0$. Let $b_n := \frac{a_n}{2^{n-1}}$. Construct the following family of sets by recursion (note that G_0 and G_{-1} are not used in the recursive step):

$$\begin{array}{ll}
G_{-1} & := \{(-\infty, 0)\} \\
F_0 & := \{[0,1]\} \\
G_0 & := \{(1, \infty)\} \\
F_{n+1} & := \{[x, m - \frac{b_{n+1}}{2}],\ [m + \frac{b_{n+1}}{2}, y] \mid [x,y] \in F_n \text{ and } m = \frac{x+y}{2}\} \\
G_{n+1} & := \{(m - \frac{b_{n+1}}{2}, m + \frac{b_{n+1}}{2}) \mid [x,y] \in F_n \text{ and } m = \frac{x+y}{2}\}
\end{array}$$

Informally F_{n+1} takes all the intervals in F_n, removes an interval of length b_{n+1} from the middle and puts the removed part in G_{n+1}. For example

$$F_1 = \{[0, \tfrac{1-b_1}{2}],\ [\tfrac{1+b_1}{2}, 1]\}$$
$$G_1 = \{(\tfrac{1-b_1}{2}, \tfrac{1+b_1}{2})\}$$
$$F_2 = \{[0, \tfrac{1-b_1-2b_2}{4}],\ [\tfrac{1-b_1+2b_2}{4}, \tfrac{1-b_1}{2}],\ [\tfrac{1+b_1}{2}, \tfrac{3+b_1-2b_2}{4}],\ [\tfrac{3+b_1+2b_2}{4}, 1]\}$$
$$G_2 = \{(\tfrac{1-b_1-2b_2}{4}, \tfrac{1-b_1+2b_2}{4}),\ \ (\tfrac{3+b_1-2b_2}{4}, \tfrac{3+b_1+2b_2}{4})\}$$

Let

$$\begin{aligned} G &:= \bigcup_{n\geq 1}(\bigcup G_n) \\ \mathcal{C} &:= \bigcap_{n\in\mathbb{N}}(\bigcup F_n) \end{aligned}$$

The following facts are easily verified from the definition.

- F_n is a set of 2^n disjoint closed intervals of length less or equal than $\frac{1}{2^n}$.
- $\bigcup F_{n+1} \subseteq \bigcup F_n$.
- $F_n \cup \bigcup_{k=1}^{n} G_k$ is a partition of $[0,1]$ for all $n \geq 1$.
- Each member of F_n is "sandwiched" between a set in G_{2p} and a set in G_{2q+1} for some $p, q \in \mathbb{Z}$.
- For all $n \geq 1$, G_n is a disjoint set of 2^{n-1} open intervals of length b_n. Hence $\lambda(\bigcup G_n) = 2^{n-1}b_n = a_n$. Since the sets $\bigcup G_n$ are pairwise disjoint we have that $\lambda(G) = \sum_{n=1}^{\infty} a_n < 1$.
- $\mathcal{C} = [0,1]\backslash G$. Therefore $\lambda(\mathcal{C}) = 1 - \sum_{n=1}^{\infty} a_n > 0$.

Example 8.37 $f : \mathbb{R} \to \mathbb{R}$ *maximal extendable but not Riemann integrable.*

$$f(x) := \begin{cases} 0 & \text{if } x \in \bigcup G_{2n} \text{ for some } n \in \mathbb{Z} \\ 1 & \text{if } x \in \bigcup G_{2n+1} \text{ for some } n \in \mathbb{Z} \\ \frac{1}{2} & \text{if } x \in \mathcal{C} \end{cases}$$

We know that $\mathcal{C}^{\mathsf{c}} \subseteq C_f$ since for all $x \in \mathcal{C}^{\mathsf{c}}$ there exists an $\varepsilon > 0$ such that f is constant in $(x-\varepsilon, x+\varepsilon)$. We claim that f is discontinuous on $\mathcal{C}$. Take $x \in \mathcal{C}$ and $\varepsilon > 0$. Find $m \in \mathbb{N}$ such that $\frac{1}{2^m} < \varepsilon$. Since $x \in \bigcup F_n$ for all n, in particular $x \in \bigcup F_m$. Therefore, there exists an interval $E \in F_m$ such that $x \in E$. But the intervals in F_m have length less than $\frac{1}{2^m}$. Since E lies between a set in G_{2p} and a set in G_{2q+1}, then the neighborhood $(x-\varepsilon, x+\varepsilon)$ must contain points y and z such that $f(y) = 0$ and $f(z) = 1$. This also shows that $\mathcal{C}^{\mathsf{c}}$ is a dense subset of $\mathbb{R}$. Hence C_f is dense, f_c exists and is given by,

$$f_c(x) := \begin{cases} 0 & \text{if } x \in \bigcup G_{2n} \text{ for some } n \in \mathbb{Z} \\ 1 & \text{if } x \in \bigcup G_{2n+1} \text{ for some } n \in \mathbb{Z} \\ [0,1] & \text{if } x \in \mathcal{C} \end{cases}$$

By Corollary 8.32 $(f_c)_m$ is maximal and clearly extends f. But we know that $\lambda((C_f)^{\mathsf{c}}) = \lambda(\mathcal{C}) > 0$, therefore f is not Riemann integrable.

Example 8.38 *Compare with the previous example.*

$$h(x) := \begin{cases} 0 & \textit{if } x \notin \mathcal{C} \\ \frac{1}{2} & \textit{if } x \in \mathcal{C} \end{cases}$$

Then h is not continuous but $C_h = \mathcal{C}^{\mathsf{c}}$ is dense and $h_c(x) = 0$ for all x.

The above construction is easier to visualize by comparing it with the classic construction of the Cantor ternary set. We recall the construction of $\mathcal{C}$ using base 3 expansions of the numbers in $[0, 1]$.

$$\mathcal{C} = \{0.a_1 a_2 a_3 \cdots \mid a_n \in \{0, 2\}\}$$

Therefore if $x = 0.a_1 a_2 a_3 \cdots \notin \mathcal{C}$, then there exists a minimum k such that $a_n \in \{0, 2\}$ for $n < k$ and $a_k = 1$. The converse is not true since some numbers have two ternary expansions, e.g. $\frac{1}{3} = 0\overline{2} = .1\overline{0}$ and $\frac{2}{3} = .2\overline{0} = .1\overline{2}$. Define for all $k \in \mathbb{N}_+$

$$\begin{aligned} G_0 &:= [-\infty, 0) \cup (1, \infty] \\ G_k &:= \{0.a_1 a_2 a_3 \cdots \notin \mathcal{C} \mid a_n \in \{0, 2\} \text{ for } n < k \text{ and } a_k = 1\} \end{aligned}$$

Hence $G_1 = (\frac{1}{3}, \frac{2}{3})$, $G_2 = (\frac{1}{9}, \frac{2}{9}) \cup (\frac{7}{9}, \frac{8}{9})$, $\ldots$ is a sequence of disjoint open sets such that $\mathcal{C}^{\mathsf{c}} = \bigcup_{n \in \mathbb{N}} G_n$.

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List of Symbols

Symbol		**Defined in**
$:=$	Assignation	Sec. 1.2
$\equiv$	Equality by definition	
$\dot{-}$	Truncated substraction	Sec. 7.7.3
$\prec$	Stochastic order	Sec. 7.5
$\trianglelefteq$		Sec. 7.6
$\square(a,b)$	The set $\{\mathbf{x} \in \mathbf{I}\mathbb{R} \mid \mathbf{x} \subseteq (a,b)\}$	Sec. 8.3
$\underline{\mathbf{x}}$	Left endpoint of interval $\mathbf{x}$	
$\overline{\mathbf{x}}$	Right endpoint of interval $\mathbf{x}$	
$\lvert\mathbf{x}\rvert$	Length of interval $\mathbf{x}$	
$\mathcal{A}(\mathcal{P})$	**Algebra** generated by the paving $\mathcal{P}$	Sec. 1.7
$a(X)$	**Antispectrum** of X	Sec. 5.4
$(A(X), \sqsubseteq)$	Lattice representation of X	
$atm(\Gamma)$	Set of **atoms** of algebra Γ	Sec. 4.3
$\mathcal{B}X$	**Borel** σ-algebra of the topological space X	Sec. 1.7
$(X, \mathcal{G}, \mathcal{G}^*)$	**Bitopological** space	Def. 3.43
$(\mathbf{B}X, \sqsubseteq)$	Space of formal **balls** of X	Sec. 5.3.5
$B(x,r)$	Open **ball** of radius r and center x	Sec. 1.12
$\mathfrak{BR}_+(X)$	Finite Borel regular outer measures	Sec. 7.7.2
$\mathfrak{BR}^1(X)$	Borel regular outer probability measures	Sec. 7.7.2
$C(x,r)$	Closed **ball** of radius r and center x	Sec. 5.3.5
C_f	Set of continuity points of f	Sec. 8.2
$cl(B)$	**Closure** of set B	Sec. 1.4
B^{c}	**Complement** of the set B	Sec. 1.2
$Cres(X)$	Set of **crescents** of topological space X	Sec. 1.11
D_f	Set of **discontinuities** of f	Sec. 6.2.1
$d_{\mathbf{V}}$	Sup-distance for valuations	Sec. 7.7.3
$A \backslash B$	**Difference** of sets A and B	Sec. 1.2
δ_x	Point (**Dirac**) valuation based at x	Sec. 1.10
$Y = \bigcup^{\uparrow} \mathcal{P}$	Y is the **directed union** of paving $\mathcal{P}$	Sec. 1.2
$\mathcal{P} \uparrow Y$	Y is the **directed union** of paving $\mathcal{P}$	
$\sum \mathcal{P}$	**Disjoint** union of paving $\mathcal{P}$	Sec. 1.2
$\underline{\mathbf{f}}, \overline{\mathbf{f}}$		Sec. 8.3

Symbol		**Defined in**
f_c, $con(f)$	Maximal continuous representation of f	Sec. 8.2
$\mathbf{f}_c$	Maximal continuous representation of $\mathbf{f}$	Sec. 8.5
$\mathbf{f}_m$, $mon(\mathbf{f})$	Maximal monotone extension of $\mathbf{f}$	
F_r, $res(F)$	Restriction of an interval function F	
f_E, f^E	Semi-continuous **envelopes** of f	Sec. 6.2.1
$\mathbf{f}_E := [\underline{\mathbf{f}}_E, \overline{\mathbf{f}}^E]$	Interval envelope of an interval function $\mathbf{f}$	Sec. 8.3
$f^\square := [f_E, f^E]$	Interval envelope of a real function f	
f_D, f^D	Greatest continuous **extensions** of f	Sec. 6.2.2
f_*, $f_{(*)}$, $f_{[*]}$		Sec. 6.4
f^*, $f^{(*)}$, $f^{[*]}$		
$\mathbb{F}_1^2(\_ \mid y)_{y\in X_2}$	**Factorization**	Def. 4.24
$Y = \bigcap_\downarrow \mathcal{P}$	Y is the **filtered intersection** of $\mathcal{P}$	Sec. 1.2
$\mathcal{P}{\downarrow}Y$	Y is the **filtered intersection** of $\mathcal{P}$	
$A \subseteq_f B$	A is a **finite subset** of B	Sec. 1.2
$\mathcal{F}$, $\mathcal{F}X$	Paving of closed sets of space X	Sec. 1.4
$\mathcal{G}$, $\mathcal{G}X$	Paving of open sets of space X	
$\mathcal{G}\restriction_Y$	Subspace topology on Y	
$\mathcal{G}_{cc}$	Cocompact topology	
$\mathcal{G}_p$	Patch topology	
$\mathcal{G}_A$	Alexandroff topology	Sec. 1.6
$\mathcal{G}_\sigma$	Scott topology	
$\mathcal{G}_u$	Upper topology	
$\mathcal{G}_l$	Lower topology	
$\mathcal{G}_\lambda$	Lawson topology	
$\mathcal{G}_V$	Vietoris topology	Sec. 5.3.2
$\mathcal{G}_{lV}$	Lower Vietoris topology	
$\mathcal{G}_{uV}$	Upper Vietoris topology	
$\square G$	Upper Vietoris basic open	
$\lozenge G$	Lower Vietoris basic open	
∇H	Projection of open set H	
$\mathcal{G}_{c-o}(X,Y)$	Compact-open topology	Sec. 6.2.4
$\langle K \to H\rangle$	Compact-open subbasic open	
$\mathcal{G}_{is}(X,Y)$	Isbell topology	
$\langle \mathcal{U} \leftarrow H\rangle$	Isbell subbasic open	
$\mathcal{G}_{pw}(X,Y)$	Pointwise convergence topology	
$\langle x \to H\rangle$	Pointwise subbasic open	
$\mathcal{G}_{va}(\mathfrak{M}X)$	Vague topology	Sec. 7.2

Symbol		Defined in
$\mathcal{G}_w(\mathfrak{M}X)$	Weak topology	Sec. 7.2
$\to_w$	Convergence in the weak topology	
$\mathfrak{I}$	$[0,1]$ as bitopological space	Sec. 3.7
I^∞	Hilbert cube	Sec. 1.12
$\mathbf{I}[a,b]$	The set $\{\mathbf{x} \in \overline{\mathbb{R}} \mid \mathbf{x} \subseteq [a,b]\}$	Sec. 8.3
Id	**Identity** function	Sec. 1.2
fA, $f[A]$	**Image** of the set A under function f	Sec. 1.2
$int(B)$	**Interior** of set B	Sec. 1.4
$\mathbb{I}_U$	**Indicator** function of set U	Sec. 1.7
$f^{-1}A$	**Inverse image** of set A under f	Sec. 1.2
$\mathbf{I}\overline{\mathbb{R}}, \mathbf{I}\mathbb{R}$	**Interval** domains	Sec. 8.3
$j^\natural$		Sec. 5.3.1
$\mathcal{G} \vee \mathcal{H}$	**Join** of topologies $\mathcal{G}$ and $\mathcal{H}$	Sec. 1.4
$\mathcal{K}$, $\mathcal{K}X$	Paving of compact sets of space X	Sec. 1.4
$\bigsqcup^\uparrow D$	**Least** upper bound of directed set D	Sec. 1.3
$\mu(f)$	**Lebesgue** integral of f wrt μ	Sec. 1.7
$\int \mathbf{f} d\mu$	Interval Lebesgue intergral	Sec. 8.4
$\mathcal{L}(\mu)$	Paving of **Lebesgue** μ-measurable sets	Sec. 1.7
$(\Lambda X, \sqsubseteq)$	Space of formal **lenses** of X	Def. 5.26
$\varprojlim \mu_i$	Projective **limit**.	Sec. 4.2
$\liminf_n E_n$	**Limit** inferior of a sequence of sets (E_n)	Sec. 1.2
$\limsup_n E_n$	**Limit** superior of a sequence of sets (E_n)	Sec. 1.2
$\downarrow B$	**Lower** set of B	Sec. 1.3
$\mathfrak{M}(X)$	Borel **measures** on X	Sec. 1.7
$\mathfrak{M}^1(X)$	Probability **measures** on X	
$\mathfrak{M}_+(X)$	Finite Borel **measures** on X	
$\mathfrak{M}(X;\tau)$	τ-smooth **measures** on X	
$\mathfrak{M}(X;R)$	Inner regular **measures** on X	
$\mathfrak{M}(X;F)$	$\mathcal{F}$-regular **measures** on X	
μ_*	Inner μ-**measure**	
μ^*	Outer μ-**measure**	
$\mathcal{M}(\mu)$	Paving of μ-**measurable** sets	
μ^1, $\overline{\mu}^1$		Sec. 7.4
$\nu^\perp$, $\overline{\nu}^\perp$		
$\max(P)$	Set of **maximal** elements of preorder P	Sec. 1.3
$\min(P)$	Set of **minimal** elements of preorder P	
$\mathbb{N}$	Set of **natural** numbers	Sec. 1.2

Symbol		**Defined in**
$\overline{\mathbb{N}}$	Extended **natural** numbers: $\mathbb{N} \cup \{\infty\}$	Sec. 1.2
$\mathbb{N}_+$	Positive **natural** numbers: $\{1, 2, \dots\}$	
$\overline{\mathbb{N}}_+$	The set $\{1, 2, \dots, \infty\}$	
$\mathbb{N}^{(\mathbb{N})}$	Set of multi-indices	Sec. 4.3
$\mathcal{N}(B)$	Paving of open **neighborhoods** of a set B	Sec. 1.4
$Net(y)$		Sec. 8.2
$\sqsubseteq^{op}$	**Opposite** of preorder $\sqsubseteq$	Sec. 1.3
$\mathbb{P}_1^2(\_ \mid y)_{y \in X_2}$	Disintegration	Def. 4.18
$\mathbb{P}_S^T(\_ \mid \_)$	Regular conditional probability distribution	Sec. 4.4
P^σ, P^λ	P topologised with $\mathcal{G}_\sigma$, $\mathcal{G}_\lambda$	Sec. 1.6
$\mathbf{P}^1 X$	Normalised **probabilistic** powerdomain of X	Sec. 1.10
$\mathbf{P}^r X$	r-normalised valuation space of X	Sec. 6.3
π-system	pi-system	Sec. 1.7
π_{BA}, π_α	Natural projection functions	Sec. 1.2
$\pi_i[\mu]$	Marginal measures	Sec. 1.8
$\otimes_\alpha \Gamma_\alpha$	**Product** σ-algebra	Sec. 1.8
$\otimes_\alpha \mathcal{G}_\alpha$	**Product** topology	Sec. 1.4
$(X, \mathcal{H}, \leq)$	**Pospace**	Sec. 1.6
$\mathcal{U}_\leq$	Upper open sets of a pospace	
$\mathcal{L}_\leq$	Lower open sets of a pospace	
2^Ω	**Power** set	Sec. 1.2
2^Ω_f	Set of finite subsets	
$\mathbb{Q}$	Set of **rational** numbers	
$\mathcal{Q}$, $\mathcal{Q}X$	Paving of compact saturated sets of X	Sec. 1.4
ρ_d	Lévy-Prohorov metric	Sec. 7.7.1
γ_d		
$\mathbb{R}$	Set of **real** numbers	Sec. 1.2
$\overline{\mathbb{R}}$	Extended **real** numbers: $[-\infty, \infty]$	
$\mathbb{R}_+$	Non-negative **real** numbers: $[0, \infty)$	
$\overline{\mathbb{R}}_+$	The set $[0, \infty]$	
$R_\sqsubseteq$	Preorder $\sqsubseteq$ seen as a binary relation	Sec. 1.3
R^l, R^u	**Riemann-Edalat** integrals	Sec. 6.4
$R(\mathbf{f}, \mu)$	Interval Riemann-Edalat integral	Sec. 8.4
$RPt(\mathcal{G})$	Proximal filters of $\mathcal{G}$	Sec. 5.4
$\upharpoonright_B$	**Restriction** of a function to set B	Sec. 1.2
$\Gamma \upharpoonright_\Omega$	**Restriction** of a σ-algebra to set Ω	Sec. 1.7
S_f	Set of maximal continuity points of f	Sec. 8.2

Symbol		Defined in
$sat(B)$	**Saturation** of set B	Sec. 1.4
$(L, \sqsubseteq, \wedge)$	**Semilattice**	Sec. 5.2
$Spec\ L$	**Spectrum** of semilattice L	
$spt(\mu)$	Support of μ	Sec. 7.7.2
$(\mathbb{S}X, \mathcal{G}_\sigma\mathbb{S})$	**Sierpinski** lattice	Sec. 5.3.1
σ-algebra		Sec. 1.7
$\sigma(\mathcal{P})$	σ-algebra generated by the paving $\mathcal{P}$	
$\sigma(f)$	σ-algebra induced by the function f	
$\sqsubseteq_s$	**Specialisation** order	Sec. 1.6
T_d	Hutchinson metric	Sec. 7.7.2
$\mathcal{T}(\mathcal{P})$	**Topology** generated by the paving $\mathcal{P}$	Sec. 1.4
$tr(\mathcal{P} \mid \Omega_0)$	**Trace** of paving $\mathcal{P}$ on set Ω_0	Sec. 1.2
$\mathcal{P} \cap \Omega_0$	**Trace** of paving $\mathcal{P}$ on set Ω_0	
τ-smooth		Sec. 1.7
$\mathbf{U}X$	**Upper** space of X	Sec. 5.3.2
$\mathbf{U}_\perp X$	Pointed **upper** space of X	
$\mathcal{U}(\Gamma)$	**Universally** measurable sets of Γ	Sec. 1.7
$\uparrow B$	**Upper** set of B	Sec. 1.3
V^{+l}	Lower non-negative **valuation integral**	Sec. 6.3
V^l	Real-valued **valuation integral**	Sec. 6.5
V^{+u}, V^u	Upper **valuation integrals**	Sec. 6.5
$V(\mathbf{f}, \nu)$	Interval valuation integral	Sec. 8.4
$\mathbf{V}X$	Set of continuous **valuations** on X	Sec. 1.10
$\mathbf{V}_+X$	Finite **valuation** space of X	Sec. 6.3
$\ll$	**Way** below (approximates)	Sec. 1.3
$\ll_\mathcal{G}$	Way below relation in the lattice $(\mathcal{G}, \subseteq)$	Sec. 1.1
$\twoheaduparrow B$	Points approximated by an element of B	Sec. 1.3
$\twoheaddownarrow B$	Set of points way below an element of B	
X^{cc}, X^p	X topologised with $\mathcal{G}_{cc}$, $\mathcal{G}_p$	Sec. 1.4
$X^\bullet = (X, \mathcal{G}_\bullet)$	X topologised with $\mathcal{G}_\bullet$	Sec. 1.1
$[X \to Y]$	Set of continuous functions from X to Y	Sec. 1.4
$\lceil X \to Y \rceil$	**Bounded** functions in $[X \to Y]$	Sec. 6.2
$[X \to Y]^\bullet$	$[X \to Y]$ topologised with $\mathcal{G}_\bullet(X, Y)$	Sec. 6.2.4
$\mathbb{Z}$	Set of **integer** numbers	Sec. 1.2

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